FUNCTIONS, RELATIONS, AND TRANSFORMATIONS

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HOLT, RINEHART AND WINSTON OF CANADA, LIMITED
Toronto • Montreal

Printed in Canada

1 2 3 4 5 — 70 69 68 67 66

PREFACE

The reader has studied *relations* and *functions* at a more junior level; the further study of these two fundamental concepts is the dominant theme of this volume.

Chapter 1 is devoted to logical reasoning. The informal methods of proof used in earlier courses are organized and placed on a more rigorous basis. The symbols that may be used are reserved for the last section of the chapter so that the instructor is able to make his own decision with regard to their use. This chapter may be studied as a unit before the rest of the volume or at any other convenient place. Alternatively, the instructor may wish to treat the topics separately as they become applicable to other portions of the book. The authors do not see any need in this volume for a time-consuming treatment of logic.

Relations and functions are considered in Chapter 2, with a new emphasis on a function as a *mapping*. The inverse of a function is introduced.

Chapters 3 and 4 deal with the conic sections as second-degree relations in the plane. In the first of these two chapters, the graphs of the conics are constructed, following a consideration of the defining equations. In each case, a study of intercepts, domain, range, and symmetry reduces the labour involved in plotting. The second of the chapters devoted to the conics develops their equations from geometric definitions. Equations of tangents to particular conics are determined and a supplementary section considers asymptotes of the hyperbola.

The next pair of chapters deals with the trigonometric functions. Considerable practice with the graphs of the sine and cosine functions illustrates the treatment of amplitude, period, phase angle, and phase shift. The development of various formulae, including those for the sine, cosine, and tangent of the sum (or difference) of two real numbers, permits the student to prove a wide variety of trigonometric identities.

The final three chapters contain a study of translations (Chapter 7), rotations (Chapter 8), and reflections and dilatations (Chapter 9).

In each of these chapters, the student finds the images of points, lines, and conics under the various mappings. Invariance of length and angle under each mapping is noted, if present.

Throughout the book, supplementary sections and also paragraphs or brief notes supplementary in nature have been included where necessary for mathematical completeness.

At the end of each exercise, harder questions or those dealing with supplementary material are numbered in red. Each chapter concludes with a concise summary of the material covered, followed by a review exercise.

The authors wish to thank Messrs. J. Cisarchuk, C. Arthur, A. Dyck, and L. James of the University of Waterloo for their assistance in the preparation of the answers to the exercises.

H.A.E.
K.D.F.
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CONTENTS

Chapter 1

LOGICAL REASONING

1.1. Sentences and Statements

It has been stated that the laws of logic are simply the laws of English grammar (or the laws of French grammar if one happens to be using the French language). While there is considerable truth in this assertion, it is an over-simplification. However, weakness in the use of language does lead to errors in reasoning and hence to errors in mathematics.

The type of sentence used in logic is the *declarative sentence.* This is a sentence that declares or states something. Some examples follow.

(a) John is weak in mathematics.

(b) All boys dislike homework.

(c) $4 + 5 > 3$.

(d) The world will end on January 16, 2176.

Interrogative sentences such as, "Did you manage to solve question (8)?" are not used in logic. Imperative sentences such as, "Report to the Principal's office", are not used in logic. Exclamatory sentences such as, "Hurrah, it is now time for the mathematics lesson!", are not used in logic.

DEFINITION. *A statement* is a meaningful declarative sentence that is either true or false, but not both.

If a sentence is meaningless, it cannot be classified as a statement. However, what is meaningless to one person may be meaningful to someone else. The sentence "The base angles of an isosceles triangle are equal" would be meaningless to someone who had never studied geometry. The sentence "*La plume de ma tante est sur le bureau de mon oncle*" would be meaningless to someone who knew no French.

In addition, the truth or falsity of a sentence will depend on the person making the judgment or on the time and circumstances in which the statement was made.

The sentence "The world will end on January 16, 2176" is certainly either true or false, but we will never know which. The sentence "John is weak in mathematics" may be judged true by his teacher since John made only 60% on the last test. However, John's friend, Bob, may judge it to be false since Bob made only 30%!

Examples.

(1) "Fish live in water" is a true declarative sentence (a statement).

(2) "$4 + 3 = 12$" is a false declarative sentence (a statement).

(3) "All hignals are progals" is a meaningless sentence (not a statement). (Note that it would cease to be meaningless if hignals and progals had been previously defined.)

(4) "He enjoys mathematics" is a sentence that is neither true nor false (not a statement).

(5) "$x + 7 = 10$" is a sentence that is neither true nor false (not a statement).

Examples 4 and 5 are known as *open sentences.* The pronoun "he" in Example 4 prevents us from being able to state whether the sentence is true or false. The pronoun is a *variable* that may be replaced by the names of suitable elements from a *replacement set.* Such a set is the *domain* of the variable. In Example 4, the domain of the variable might be the set of all boys in your mathematics class. We should then presume that the open sentence would become a true statement whenever "he" is replaced by the name of any member of the replacement set!

In Example 5, the variable is x and the domain of the variable might be the set of natural numbers. Only when x is replaced by 3 does the open sentence become a true statement. For all other, elements of the domain it is a false statement.

EXERCISE 1.1

Answer the following questions about each of the sentences below.

(a) Is it a declarative sentence?
(b) Is it a statement? (If so, is it true or false?)
(c) Is it an open sentence? (If so, state a possible domain.)

1. All rational numbers are integers.
2. How old are you?
3. He is Prime Minister of New Zealand.
4. $7 + 5 = 12$.
5. $6 > 2$.

6. For all $x \in Re$, $x^2 - 1 = (x - 1)(x + 1)$.
7. For all $x \in Re$, $x^2 - 1 = 15$.
8. Some boys dislike football.
9. All boys dislike football.
10. No boys dislike football.
11. Do questions (1) to (15) for homework.
12. Alternate angles are equal.
13. Some alternate angles are equal.
14. Parallel lines are coplanar.
15. She enjoys dancing.
16. Why did you not do your homework?
17. Is $2 + 3 = 5$?
18. A circle is the set of points in a plane that are a constant distance from a given point in the plane.
19. $2x + 3 = 12$.
20. For some $x \in Re$, $\frac{x}{0} = 1$.
21. For all $x \in Re$, $\frac{x}{0} = 1$.
22. For some $x \in Re$, $\frac{0}{0} = x$.
23. For all $x \in Re$, $\frac{0}{0} = x$.
24. The Yukon is in Europe.
25. $2n + 1$ is an odd number.

1.2. Definitions

A definition is not a statement since it is neither true nor false. In a definition, we give a name to a particular subset of a known set. When we examine the set of all quadrilaterals, we notice that special properties apply to quadrilaterals in which the opposite sides are parallel. Since we find that this particular subset of the set of all quadrilaterals occurs frequently in our investigations, it becomes cumbersome to be constantly saying, "A quadrilateral with opposite sides parallel".

Consequently we decide to invent a special name for such quadrilaterals and so the word "parallelogram" can be used in place of the description. Note that we are following this procedure for convenience only and that to say "A parallelogram is a quadrilateral with opposite sides parallel" merely informs others of the title we are giving to this particular subset of the set of all quadrilaterals. The sentence is neither true nor false but is only an agreement to use a particular name. Hence a definition is not a statement. However, a definition must conform to certain characteristics.

(a) The term being defined must be named in the sentence.

(b) All terms in the sentence, except the term being defined, should be known terms.

(c) The term being defined is placed in a known set.

(d) The characteristics that distinguish the term from other members of its set must be stated.

(e) The sentence should not contain any unnecessary or irrelevant information.

(f) The sentence should be reversible.

Let us consider the definition of an isosceles triangle: an isosceles triangle is a triangle with two equal sides. Does it satisfy the necessary prerequisites of a good definition?

(a) The term being defined, an isosceles triangle, is named in the definition.
(b) The terms "triangle", "equal", and "sides" have been previously defined.
(c) The term "isosceles triangle" is placed in the known set of all triangles.
(d) The characteristic that distinguishes an isosceles triangle from other members of the set of all triangles is stated.
(e) The sentence contains no unnecessary information. For instance, it does not state that an isosceles triangle has two equal angles. While this fact is true, it is unnecessary information, since it is a consequence of the definition and can be proved from the information given. However, it should be pointed out that this property of not giving unnecessary information may not always be possible and, while desirable, it is not so vital as the other points.
(f) The reversed sentence "A triangle with two equal sides is an isosceles triangle" is an equivalent sentence.

Example. Criticize the following definitions.

(a) A square is a rectangle with all four sides equal.
(b) A splot is an ink-blot which covers an area of more than 10 square inches.
(c) A turkey is an animal with two legs.

Solution:

(a) The definition gives unnecessary information. Since a rectangle has opposite sides equal and the square belongs to the set of rectangles, it is only necessary to state that two adjacent sides are equal.

(b) Assuming that we know the meaning of ink-blot, area, and 10 square inches, this sentence fulfils all the requirements of a definition.

(c) This "definition" places a turkey in the set of all animals but, since many other members of the set of all animals possess the quality of having two legs, it fails to distinguish between the turkey and these other members. In addition the sentence is not reversible. This defect is a natural consequence of the first criticism.

EXERCISE 1.2

1. Criticize the following definitions.
   (a) A circle is a plane figure enclosed by one line.
   (b) An even integer is an integer that is twice an odd integer.
   (c) A chord of a circle is a line segment joining any two points on the circle.
   (d) A cowraffe is a cow with a neck five feet long.
   (e) A quadratic polynomial is $2x^2 - 3x + 7$.
   (f) A set is a collection of objects.
   (g) A point is a position which has no dimensions.
   (h) A hignal is a progal with two left feet.
   (i) A Canadian is a person who is a citizen of Canada.
   (j) A rational number is a number that is not irrational.

2. Try to give suitable definitions for each of the following.
   (a) An even integer
   (b) A rational number
   (c) A quadratic polynomial
   (d) Happiness
   (e) School
   (f) The United Nations

3. In the following definitions, state which terms must have been previously defined or accepted as undefined.
   (a) A line segment is the set of points consisting of two points on a line and all those points on the line lying between these two points.
   (b) A chord of a circle is a line segment joining any two points on the circle.
   (c) A Canadian is a person who is a citizen of Canada.
   (d) A complex number is a number of the form $a + bi$ where $a, b \in Re$ and $i^2 = -1$.
   (e) A triangle is a polygon of three sides.

1.3. Logical Connectives (And, Or)

Statements may be combined by means of connectives of which the two principal ones are *and* and *or*. We shall represent simple statements by means of lower case letters p, q, etc. Note that the equality sign is being used here in the sense of "represents".

If p = *All triangles are isosceles,* and q = *The sum of two odd integers is an even integer,* then p *and* q represents the compound statement *All triangles are isosceles and the sum of two odd integers is an even integer*; p *or* q represents the statement *All triangles are isosceles or the sum of two odd integers is an even integer.* The first compound statement is the *conjunction* of the two original statements and the second is the *disjunction* of the two original statements. We shall examine each of these in turn.

Conjunction.

The word *and* placed between two statements p, q produces a new statement p *and* q called the *conjunction* of p and q.

If p = *All triangles are isosceles,* and q = *The sum of two odd integers is an even integer,* then p is false and q is true. The conjunction p *and* q or *All triangles are isosceles and the sum of two odd integers is an even integer* is also false. For a conjunction to be true, both component statements must be true. If one or both component statements are false, the conjunction is false.

Example. Determine the truth or falsity of the compound statement p *and* q in (1) to (3). If it is false, give a reason.

(1) $p = 3 + 1 > 2.$
q = All similar triangles are congruent.

(2) $p = 3 + 1 > 2.$
q = All congruent triangles are similar.

(3) $p = (x + y = xy)$ for all $x, y \in N$.
q = Alexander the Great was a Roman citizen.

Solution:

(1) p *and* q is false since q is false.

(2) p *and* q is true since both p and q are true.

(3) p *and* q is false since both p and q are false.

Disjunction.

The word *or* placed between two statements p, q produces a new statement p *or* q called the disjunction of p and q.

We note that *or* is used mathematically in the inclusive sense; p *or* q means *either* p *or* q *or both.* If we wish to use the exclusive *or*, then we must state p *or* q *but not both.*

In ordinary English usage, the statement "I should make over 75% in mathematics or physics" does not exclude the possibility of over 75% in both. Hence, *or* is used in the inclusive sense. However, the statement "Tomorrow I will travel to Winnipeg by car or by train" means that we will use only one of the means of travel but not both. In this case, *or* is used in the exclusive sense. In mathematics, the inclusive *or* is always understood unless otherwise specified.

If p = *All triangles are isosceles,* and q = *The sum of two odd integers is an even integer,* the disjunction p *or* q is the statement *All triangles are isosceles or the sum of two odd integers is an even integer.* The disjunction is true since q is true. A disjunction is true whenever one or both of the component statements is true. A disjunction is false only if both component statements are false.

Example. Using the statements p, q given in the example for conjunction, determine the truth or falsity of the statement p *or* q, giving reasons.

Solution:

(1) p *or* q is true since p is true.

(2) p *or* q is true since both p and q are true.

(3) p *or* q is false since both p and q are false.

EXERCISE 1.3

For each of the following statements, determine whether the conjunction is true or false and whether the disjunction is true or false. Give a reason in each case.

1. p = 5 and 6 are consecutive integers.
   q = All right angles are equal.
2. p = London is the capital city of the United Kingdom.
   q = Dogs have four legs.
3. p = The equation $y = x^2 + 6$ determines a function.
   q = London is the capital city of Belgium.
4. p = The equation $x^2 + y^2 = 16$ is the equation of a line.
   q = The sum of two consecutive integers is an even integer.
5. p = -3 is a positive integer.
   q = π is a rational number.
6. p = Equal chords of a circle are equidistant from the centre.
   q = The line $y = 3x + 2$ has slope 3.
7. p = If $y = \sin x$, $x \in Re$, then $0 \leq y \leq 1$.
   q = All professors are absent-minded.
8. p = The eighth term of the arithmetic sequence $1, 4, 7, \cdots$ is 22.
   q = $\sqrt{2}$ is not a rational number.

9. $p =$ An acute angled triangle is a triangle in which one of the angles is an acute angle.
   $q =$ The seventh term of the geometric sequence $1, 2, 4, \cdots$ is 64.

10. $p = \left(\dfrac{x^2 - 4}{x - 2} = x + 2\right)$ for all $x \in Re$.
    $q = 4 + 5 > 3$.

11. $p =$ The system of equations $2x - y = 7$ and $6x - 3y = 8$ has a solution set consisting of one ordered pair of real numbers.
    $q =$ The solution set of the system of equations $x - 3y = 1$ and $2x + 3y = 11$ is $\{(1, 4)\}$.

12. $p =$ The moon is approximately 250,000 miles from the earth.
    $q =$ The graph of $y = 2x^2 - 3x + 6$ is a parabola.

1.4. Negation

If p is the statement *All triangles are isosceles,* then *not p* is the statement *Not all triangles are isosceles.* The new statement is the negation of p. The negation may be worded in several ways in addition to that given. Other forms would be *It is false that all triangles are isosceles,* or *It is not true that all triangles are isosceles.*

We should be careful that the word 'not' be correctly placed. The statement "All triangles are not isosceles" is not the negation of p. In this statement, the adjective "isosceles" has been negated instead of the whole statement. The statement "Not all triangles are isosceles" implies that some triangles may be isosceles, while the statement "All triangles are not isosceles" implies that no isosceles triangles exist. Some care is necessary here since in colloquial English usage the sentence might be taken to mean, "Not all triangles are isosceles". This case is one in which carelessness in English usage can lead to a false conclusion. Later, we shall see other ways of negating a statement containing the word "all".

If a statement p is true, then the statement *not p* is false. If a statement p is false, then the negation of p is true. The statement *All triangles are isosceles* is false, but the negation *Not all triangles are isosceles* is a true statement.

When p is a simple statement, the negation of p is fairly obvious. However, the negation of compound statements containing the connectives "and" or "or" requires some careful thought.

The statement *Mary is blonde and John is tall* asserts that *Mary is blonde* and *John is tall* are both true statements. For the conjunction to be true, both simple statements must be true. The negation denies that *both* are true and, therefore, asserts that *at least one* is false. The negation is *Mary is not blonde or John is not tall.*

This situation may be summarized by stating that the negation of *p and q* is *(not p) or (not q).*

The statement *Mary is blonde or John is tall* asserts that *at least one* of the statements *Mary is blonde* and *John is tall* is a true statement. For the disjunction to be true, either one or both of the simple statements must be true. To negate the disjunction, we must assert that *both* statements are false. The negation is *Mary is not blonde and John is not tall.*

The negation of p *or* q is (*not* p) *and* (*not* q).

The negation of the statement (5) *is not an even integer* is the statement (5) *is an even integer.*

In general, the negation of (*not* p) is p. The negation of *Mary is blonde* may be written in various forms.

(1) *Mary is not blonde.*

(2) *It is false that Mary is blonde.*

(3) *It is not true that Mary is blonde.*

EXERCISE 1.4

Write the negation of each of the following statements. If possible, state whether the statement or its negation is true.

1. $4 + 5 = 20$.
2. All right angles are equal.
3. The world will end on January 16, 2176.
4. The sum of two consecutive integers is an even integer.
5. -3 is a positive integer.
6. π is a rational number.
7. $3 + 1 > 2$.
8. $x^2 + y^2 = 16$ is the equation of a line.
9. $\sqrt{2}$ is not a rational number.
10. Toronto is the capital city of Canada.
11. $2 + 3 = 5$ and dogs have four legs.
12. All right angles are equal and the graph of $y = 3x^2$ is a circle.
13. $2 + 3 = 5$ or dogs have four legs.
14. All right angles are equal or the graph of $y = 3x^2$ is a circle.
15. All similar triangles are congruent or -3 is a natural number.
16. 100 is 20% of 800 and the angle in a semicircle is a right angle.
17. π is a rational number or $\sqrt{2}$ is a rational number.
18. Alexander the Great was a Roman citizen and $2x + 1 = 3$.

1.5. Conditional Statements

The statement *The base angles of an isosceles triangle are equal* is a *conditional* or *implicative* statement. To clarify this, we may rewrite the statement as follows:

If a triangle is isosceles, then its base angles are equal.

The statement is a compound statement formed by using the connective *if . . . , then . . .* to connect two simple statements. If p represents the statement *A triangle is isosceles*, and q represents the statement *The base angles are equal*, then the conditional may be written as "If p, *then* q", or as "p *implies* q". This is frequently written symbolically as $p \Rightarrow q$. Statement p is called the *antecedent* and statement q is called the *consequent* of the conditional statement.

Conditional statements may be written in various equivalent forms and it is essential that we recognize the different phrasings in reading mathematics.

We shall consider the statement *Congruent triangles are similar*. This may be written as follows: *If two triangles are congruent, then the triangles are similar.*

Another possible phrasing is *A sufficient condition for two triangles to be similar is that the triangles be congruent.*

We not that our statement also implies that if the triangles are not similar, then they cannot be congruent since congruency implies similarity.

Hence, other possible phrasings are

(1) A *necessary* condition for two triangles to be congruent is that the triangles be similar;

(2) Two triangles are congruent *only if* they are similar.

If p = *Two triangles are congruent*,
q = *The triangles are similar*,

then the possible phrasings for the conditional statement $p \Rightarrow q$ are as follows:

(a) If p, then q.
(b) q, if p.
(c) p implies q.
(d) p is sufficient for q.
(e) p, only if q.
(f) q is necessary for p.

When is a conditional statement true and when is it false? To examine this question, we shall use two simple statements.

p = *It rains tomorrow.*
q = *The game will be cancelled.*

If John makes the statement *If it rains tomorrow, then the game will be cancelled*, under what conditions is he speaking the truth? He is stating that $p \Rightarrow q$.

If p is true and q is true, then obviously John is speaking the truth. It does rain and the game is cancelled.

Suppose p is true and q is false. It does rain but the game is not cancelled. John's statement is false.

Suppose p is false and q is true. It does not rain and the game is cancelled.

John only stated that if it did rain, the game would be cancelled. He did not make any statement about what would happen if it did not rain. His statement is true.

Suppose p is false and q is false. It does not rain and the game is not cancelled. John's statement is true.

We note that a conditional statement is always considered to be true when the antecedent is false. Only when the antecedent is true and the consequent false can we consider the statement to be false.

Since the disjunction of *not* p and q, that is, (*not* p) *or* q, is also a false statement only when p is true and q is false, the conditional statement, $p \Rightarrow q$ is equivalent to the statement (*not* p) *or* q.

The statement *If it rains tomorrow, then the game will be cancelled* is equivalent to the statement, *It will not rain tomorrow or the game will be cancelled.*

The statement *If two triangles are congruent, then they are similar* is equivalent to the statement, *Two triangles are not congruent or they are similar.*

Since the negation of a disjunction is the conjunction of the negations of the simple statements, the negation of (*not* p) *or* q, and hence also of $p \Rightarrow q$, is the statement p *and* (*not* q). This also follows from the fact that an implication is false only when the antecedent is true and the consequent false. Hence, the negation is true only when the antecedent is true and the consequent false.

The negation of the statement *If it rains tomorrow, then the game will be cancelled* is the statement *It will rain tomorrow and the game will not be cancelled.*

The negation of the statement *If two triangles are congruent, then they are similar* is the statement *Two triangles are congruent but they are not similar.* Note that the word *but* is equivalent to *and* in this statement.

It must be emphasized that the negation of a conditional statement is *not* a conditional statement but is a conjunction of two statements.

EXERCISE 1.5

1. For each of the following statements p and q, write the conditional statements $p \Rightarrow q$ and $q \Rightarrow p$. State which conditional statements are true and which false.

   (a) $p = (\text{In } \triangle ABC, \angle ABC = 90°.)$
   $q = (AC^2 = AB^2 + BC^2.)$

   (b) $p = (1 + 3 = 5.)$
   $q = (2 + 3 = 6.)$

   (c) $p = (2 + 3 = 6.)$
   q = Ottawa is the capital of Canada.

   (d) $p = (\triangle ABC \equiv \triangle DEF.)$
   $q = (\triangle ABC = \triangle DEF.)$

   (e) p = Air contains 90% hydrogen.
   q = Sicily is a country in South America.

   (f) p = Quadrilateral ABCD has its opposite sides equal.
   q = Quadrilateral ABCD is a parallelogram.

   (g) p = Quadrilateral ABCD has its opposite sides equal.
   q = Quadrilateral ABCD is a rectangle.

   (h) $p = 7 + 1 > 5.$
   $q = 2 + 1 > 5.$

2. Rewrite each of the sentences in parts (a) to (h) in each of the following ways.

    (i) Using the words, "If . . ., then"

    (ii) Using the words, "sufficient condition"

    (iii) Using the words, "necessary condition"

    (iv) Using the words, "only if"

    (a) Two lines are parallel if the alternate angles formed by a transversal are equal.
    (b) Congruent triangles are equal in area.
    (c) The solution of the equation $2x + 1 = 5$ is $x = 2$.
    (d) A parallelogram is a quadrilateral in which the opposite sides are parallel.
    (e) Since $x = 0$, $xy = 0$.
    (f) $x = 0$ if $xy = 0$.
    (g) The sum of two consecutive integers is an odd integer.
    (h) A normal dog is an animal with four legs.

3. Write the negation of each of the statements in question (2) and state whether the original statement or the negation is true.

4. John was told by his father, "You may remain in school if you do your homework." John did not do his homework but his father allowed him to remain in school. Is John's father truthful?

5. The students' council of Brandex High School stated, "A deposit of \$1 is sufficient to reserve your copy of the yearbook." John only had 50¢. In view of the council's statement, could the treasurer accept John's 50¢ and reserve a copy of the yearbook for him?

6. Albert Jones is employed by the Grimm company. A sentence in his contract states, "An employee shall work overtime only if he is over 25 years of age." Albert is 32. Must he work overtime?

1.6. Converse, Inverse, and Contrapositive

If

$$p = \triangle ABC \equiv \triangle DEF\,,$$

and

$$q = \triangle ABC \,|||\, \triangle DEF\,,$$

then $p \Rightarrow q$ is the statement "If $\triangle ABC \equiv \triangle DEF$ then $\triangle ABC \,|||\, \triangle DEF$." p is the antecedent and q is the consequent of the conditional statement. By interchanging the antecedent and consequent of the statement, we form a new conditional statement, which is the *converse* of the given statement.

The converse of $p \Rightarrow q$ is $q \Rightarrow p$.
In this case, the converse statement is

If $\triangle$ABC ||| $\triangle$DEF, *then* $\triangle$ABC $\equiv$ $\triangle$DEF.

While the original conditional statement is true, the converse is false. The truth or falsity of a given conditional statement tells us nothing about the truth or falsity of the converse statement. The converse statement may also be written in the following equivalent forms.

(1) $\triangle$ABC $\equiv$ $\triangle$DEF if $\triangle$ABC ||| $\triangle$DEF.

(2) $\triangle$ABC ||| $\triangle$DEF implies that $\triangle$ABC $\equiv$ $\triangle$DEF.

(3) $\triangle$ABC ||| $\triangle$DEF is sufficient for $\triangle$ABC $\equiv$ $\triangle$DEF.

(4) $\triangle$ABC ||| $\triangle$DEF only if $\triangle$ABC $\equiv$ $\triangle$DEF.

(5) $\triangle$ABC $\equiv$ $\triangle$DEF is a necessary condition for $\triangle$ABC ||| $\triangle$DEF.

If both the original statement and the converse are true statements; that is, if $p \Rightarrow q$ and $q \Rightarrow p$ are both true, then p and q are *equivalent* statements. The statement *The base angles of an isosceles triangle are equal* may be written as *If a triangle is isosceles, then the base angles are equal.* This is a true statement, and the converse statement, *If a triangle has equal base angles, then the triangle is isosceles,* is also a true statement. We shall let p be the statement *$\triangle ABC$ is isosceles with $AB = AC$,* and q be the statement $\angle ABC = \angle ACB$. The *biconditional* statement $p \Rightarrow q$ and $q \Rightarrow p$ can be written as $p \Leftrightarrow q$. Again, this statement may be written in various forms.

(1) If p, then q, and if q, then p.

(2) If p, then q, and conversely.

(3) If q, then p, and conversely.

(4) p, if and only if q.

(5) q, if and only if p.

(6) p is a necessary and sufficient condition for q.

(7) q is a necessary and sufficient condition for p.

Sometimes, instead of $\Leftrightarrow$, the symbol $\equiv$ is used with exactly the same meaning. However, the use of the latter symbol, especially in geometrical arguments, might lead to confusion, for the same symbol is used to denote congruence of triangles.

"If and only if" is often abbreviated to "iff" in mathematical writing. Thus, (4) and (5) may be written in the forms p iff q and q iff p, respectively. (Similar abbreviations are used in mathematics written in other languages. The French *si et seulement si* becomes *ssi.*)

Since $p \Rightarrow q$ is true except when p is true and q is false, and $q \Rightarrow p$ is true except when q is true and p is false, $p \Leftrightarrow q$ will be true except when one of p and

q is true and the other false. That is, $p \Leftrightarrow q$ is a true statement if both p and q are true or if both p and q are false.

In the discussion of definitions we stated that a definition must be reversible. The definition *An isosceles triangle is a triangle with two equal sides* could be written in each of the following ways.

(1) A triangle is isosceles if and only if it is a triangle with two equal sides.

(2) A necessary and sufficient condition for a triangle to be isosceles is that the triangle have two equal sides.

From the conditional statement *If* $\triangle ABC$ *is congruent to* $\triangle DEF$, *then* $\triangle ABC$ *is similar to* $\triangle DEF$, we may form a new statement by negating both the antecedent and consequent of the original statement. Such a statement is the inverse of the original statement. The inverse of the given statement is *If* $\triangle ABC$ *is not congruent to* $\triangle DEF$, *then* $\triangle ABC$ *is not similar to* $\triangle DEF$.

The *inverse* of $p \Rightarrow q$ is $(not\ p) \Rightarrow (not\ q)$.

In the example given, the original conditional statement is true but the inverse is false. The truth or falsity of a statement does not guarantee the truth or falsity of the inverse.

By taking the same original statement and negating both the antecedent and consequent of the converse statement, we produce a new statement called the *contrapositive* of the original conditional statement. The contrapositive of the statement *If* $\triangle ABC$ *is congruent to* $\triangle DEF$, *then* $\triangle ABC$ *is similar to* $\triangle DEF$ is the statement *If* $\triangle ABC$ *is not similar to* $\triangle DEF$, *then* $\triangle ABC$ *is not congruent to* $\triangle DEF$.

In this case, both the original conditional statement and its contrapositive are true.

The *contrapositive* of $p \Rightarrow q$ is $(not\ q) \Rightarrow (not\ p)$.

A conditional statement and its contrapositive are *equivalent* statements. If the statement is true, then so is the contrapositive and if a statement is false, then so is the contrapositive.

The statement $p \Rightarrow q$ is false only if p is true and q is false. The statement $(not\ q) \Rightarrow (not\ p)$ is false only if *not* q is true and *not* p is false, that is, when q is false and p is true. These criteria are identical and so the conditional statement is equivalent to its contrapositive.

EXERCISE 1.6

1. For each of the following sentences, write

   (i) the converse, (ii) the inverse, (iii) the contrapositive

   (a) If roses are red, then violets are blue.

   (b) If $2 + 3 = 5$, then $2 + 2 < 5$.

   (c) A quadrilateral is a parallelogram if its opposite sides are parallel.

   (d) If we have no homework tonight, then I am a Martian.

(e) An integer is odd if it is the sum of two consecutive integers.

(f) Congruent triangles are equal in area.

(g) If $3x + 1 = 10$, then $x = 3$.

(h) If $3x + 1 = 10$, then $x = 4$.

(i) If Mr. Jones lives in Marseilles, then he lives in France.

(j) Opposite angles of a cyclic quadrilateral are supplementary.

2. Rewrite each of the following sentences using the "if and only if" and "necessary and sufficient" wording.

(a) Collinear points are points which lie on the same line.

(b) A Mexican is a person who is a citizen of Mexico.

(c) A rational number is a number of the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$.

(d) A radius of a circle is a line segment joining the centre to any point on the circle.

(e) An obtuse angle is an angle greater than 90° but less than 180°.

(f) A parallelogram is a quadrilateral with its opposite sides parallel.

(g) If a transversal intersects two parallel lines, then the alternate angles are equal, and conversely.

1.7. Quantifiers

The open sentence $x + 2 > 0$ cannot be said to be true or false unless we know the domain of the variable x. However, open sentences can be converted into statements by the use of *quantifiers.* The sentence

For all positive integers x, $x + 2 > 0$

is a true sentence and hence, a statement.

The sentence

For all integers x, $x + 2 > 0$

is a false sentence and hence, a statement.

Equivalent ways of wording the first sentence are

(1) *For any positive integer* x, $x + 2 > 0$,

(2) *For each positive integer* x, $x + 2 > 0$,

(3) *For every positive integer* x, $x + 2 > 0$.

The statement *For all integers* x, $x + 2 > 0$ is false since -5 is a member of the domain and $-5 + 2 \not> 0$. However, the statement *For some integers* x, $x + 2 > 0$ is a true statement. This may be reworded in the following equivalent ways.

(1) *There exists an integer* x *such that* $x + 2 > 0$.

(2) *For at least one integer* x, $x + 2 > 0$.

The words *all, any, some, there exists* give us an idea of quantity and so are known as *quantifiers.*

For all (and its equivalents) is known as the *universal quantifier.*

There exists (and its equivalents) is known as the *existential quantifier.*

The word *no* is also used as a quantifier but it is possible to replace it by *all.*

The statement *No boys are silly* is equivalent to the statement *All boys are not silly.* We should be careful here since a common error in English usage is to state that this is equivalent to the statement *Not all boys are silly.* A close examination of the two statements should show the difference.

(1) All boys are not silly.

(2) Not all boys are silly.

Sentence (1) states that *each* and *every* boy is not silly.

Sentence (2) states that *some* boys or *at least one* boy is not silly.

If we wish to negate a statement containing a quantifier, we must consider both the quantifier and the part of the statement following the quantifier.

The statement *For all* $x \in N$, $x + 1 = 4$ affirms that for all natural numbers x, $x + 1 = 4$. To deny this assertion is to say that for at least one natural number x, $x + 1 \neq 4$. If there is at least one natural number x such that $x + 1 \neq 4$, our original assertion is false and the negation is true.

In general, the quantifier *all* (and its equivalents) is negated by replacing it by the quantifier *some* or its equivalents, and then negating the second part of the statement.

The statement *Some men are bad drivers* asserts that at least one man is a bad driver. For this statement to be false, it would be necessary to assert that the statement *All men are good drivers* is true. The quantifier *some* is negated by replacing it by the quantifier *all,* and then negating the remainder of the sentence.

Examples.

1. Statement: *All angles are right angles.*
   Negation: *Some angles are not right angles.*

2. Statement: *For some positive integer* x, $x + 1 = 5$.
   Negation: *For all positive integers* x, $x + 1 \neq 5$.

3. Statement: *No boys are silly.*
   Equivalent: *All boys are not silly.*
   Negation: *Some boys are silly.*

4. Statement: *Not all boys are silly.*
   Equivalent: *Some boys are not silly.*
   Negation: *All boys are silly.*

Alternative ways of wording the negation of example (1) are:

There exist angles which are not right angles.

There is at least one angle which is not a right angle.

Not all angles are right angles.

Alternative wordings for the negation of example (2) are:

For every positive integer x, $x + 1 \neq 5$.

For each positive integer x, $x + 1 \neq 5$.

For no positive integer x is $x + 1 = 5$.

In most mathematical work, quantifiers are understood rather than stated. However, when precision is necessary or when doubt is possible, the quantifier must be used.

When we state *The base angles of an isosceles triangle are equal,* we understand the quantifier *all* and the statement would read *For all isosceles triangles T*, the *base angles* of T *are equal.*

When we state $a(a + b) = a^2 + ab$, we are stating *For all numbers a and b*, $a(a + b) = a^2 + ab$.

When we write $2x - 1 = 5$, we mean *There exists a number x such that* $2x - 1 = 5$ or *For some numbers* x, $2x - 1 = 5$.

EXERCISE 1.7

1. Use the quantifiers *all* or *some* or their equivalents to make each of the following a true statement.
   (a) $(x - 1)^2 = x^2 - 2x + 1$.
   (b) $3x + 2 = 17$.
   (c) $|x| = x$.
   (d) $\sqrt{x^2 + 9} \neq x + 3$.
   (e) The opposite sides of a parallelogram are equal.
   (f) No triangles are parallelograms.
   (g) Not all triangles are isosceles.
   (h) $x + 2x = 3x$.
   (i) The opposite angles of a cyclic quadrilateral are supplementary.
   (j) Not all rectangles are squares.

2. Negate each of the following statements.
   (a) For all $x \in Re$, $|x| = x$.
   (b) For some $x \in N$, $x^2 = 2$.
   (c) Every rational number is a real number.
   (d) For all $n \in N$, $n^2 - n + 41$ is a prime integer.
   (e) All triangles are not isosceles.
   (f) Not all triangles are isosceles.
   (g) Not all students are clever.
   (h) All students are not clever.
   (i) No students are stupid.
   (j) There exist numbers x and y such that $(x^2 - y^2) = (x - y)^2$.
   (k) For at least one $x \in Re$, $x^2 = -4$.
   (l) For any positive integer x, $x + 4 < 10$.

1.8. Principles of Proof

In proving any mathematical theorem, we show that the truth of the theorem follows by *logical inference* from statements which we have previously proved or accepted as true.

Statements accepted as true are the axioms or postulates that are the basis of the mathematical system. For example, in geometry we state as one of the axioms that, corresponding to any two points, there is one and only one line.

In proving theorems, we make use of six logical principles. These are based on the work of the preceding sections of this chapter.

(1) *Principle of Detachment*

If $p \Rightarrow q$ is true (either previously proved or accepted as an axiom), and p is true, then q is true. The statement $p \Rightarrow q$ is the *major premise*, p is the *minor premise*, and q is the *conclusion*. An example follows.

Major premise:	*If x is even, then x is divisible by* 2.	$p \Rightarrow q$.
Minor premise:	*x is even.*	p.
Conclusion:	*x is divisible by* 2.	$\therefore q$.

One of the commonest errors in logical reasoning is to assume that if $p \Rightarrow q$ is true and q is true, then p is true. Here we are assuming the converse and from Section 1.6 we know that the converse of a true statement is not necessarily a true statement.

Major Premise:	*If John lives in Calcutta, then he lives in Bengal.*	$p \Rightarrow q$.
Minor Premise:	*John lives in Bengal.*	q.
Conclusion:	*John lives in Calcutta.*	$\therefore p$.

The conclusion may be true or false. The reasoning is faulty and our conclusion cannot be deduced from the premises. A *valid* argument is one in which the premises imply the conclusion. If an argument is not valid, we call it *invalid*. The argument above is invalid. We should also note that an argument is valid or invalid, but a statement is true or false. We may not speak of a true argument or a valid statement.

(2) *Principle of Equivalence*

If two statements are equivalent and one of them is true, then the other is true. If $p \Leftrightarrow q$ is true and p is true, then q is true. If $p \Leftrightarrow q$ is true and q is true, then p is true.

Major Premise:	$2x - 1 = 5$ *iff* $x = 3$.	$p \Leftrightarrow q$.
Minor Premise:	$2x - 1 = 5$.	p.
Conclusion:	$x = 3$.	$\therefore q$.

Major Premise:	$2x - 1 = 5$ *iff* $x = 3$.	$p \Leftrightarrow q$.
Minor Premise:	$x = 3$.	q.
Conclusion:	$2x - 1 = 5$.	$\therefore p$.

(3) *Principle of Disjunction*

If p or q is true and p is false, then q is true. Note that if p is true then q may be either true or false. The principle of disjunction does not guarantee which case holds.

Premises:	(i) $AB \parallel CD$ or $AB \nparallel CD$.	p *or* q.
	(ii) $AB \nparallel CD$ is false.	*not* q.
Conclusion:	$AB \parallel CD$.	$\therefore p$.

This is the logical principle which we have frequently used in the past when proving a theorem by indirect proof. We consider the various possible results and show that all but one are false. The one remaining possibility has to be true.

(4) *Principle of the Contrapositive*

If $p \Rightarrow q$ is true and q is false, then p is false. In this example we shall use the word *hypotheses* (whose singular form is *hypothesis*), instead of premises. The words are synonymous and "hypothesis" is more frequently used in mathematics.

Hypotheses:	(i) *If* $x = 3$, *then* $x + 2 \not> 4$.	$p \Rightarrow q$.
	(ii) $x + 2 \not> 4$.	*not* q.
Conclusion:	$x \neq 3$.	*not* p.

This argument uses the contrapositive, *not* $q \Rightarrow$ *not* p, of the original $p \Rightarrow q$.

A common error in reasoning is to use the inverse instead of the contrapositive. The inverse does not provide a valid argument. In the example below, the inverse gives the argument.

Hypotheses:	(i)	*If* $x = 3$, *then* $x + 2 > 4$.	$p \Rightarrow q$.
	(ii)	$x \neq 3$.	*not* p.
Conclusion:		$x + 2 \not> 4$.	*not* q.

The conclusion may be true or false. The argument is invalid since the inverse of a true statement is not necessarily a true statement.

(5) *Principle of the Syllogism*

If $p \Rightarrow q$ is true and $q \Rightarrow r$ is true, then $p \Rightarrow r$ is true.

Hypotheses:	(i)	*If* $4x+7 = x + 1$, *then* $3x = -6$.	$p \Rightarrow q$.
	(ii)	*If* $3x = -6$, *then* $x = -2$.	$q \Rightarrow r$.
Conclusion:		*If* $4x + 7 = x + 1$, *then* $x = -2$.	$p \Rightarrow r$.

(6) *Principle of Substitution*

(i) *Substitution for a variable*

If an open sentence is made into a statement by using a quantifier so that the statement is true for all elements of the domain of the variable, then substitution of a specific element of the domain for the variable will produce a true statement.

Hypothesis:	(i)	All rational numbers are real numbers.
	(ii)	$-\frac{4}{5}$ is a rational number.
Conclusion:		$-\frac{4}{5}$ is a real number.

(ii) *Substitution for statements*

If $p \Leftrightarrow q$ is a true statement, then q may be substituted for p in any statement involving p.

Hypotheses:	(i)	A parallelogram is a rectangle if and only if it has one angle a right angle.
	(ii)	A parallelogram with one angle a right angle has equal diagonals.
Conclusion:		A rectangle has equal diagonals.

In all proofs, each step must follow from one of these principles of logic. While we will not normally state which principle we are using in reaching a conclusion, we should at all times be prepared to justify the steps in a proof. The only justification is that we have used correctly one of these logical principles.

As an example of the use of the principles of logic we will consider the following set of axioms. These are statements assumed to be true as the basis of a given discussion. Any conclusion reached is arrived at by valid reasoning using the principles of logic, and if our axioms are accepted as being true then our conclusion must also be a true statement.

Axiom 1. *All students work hard.*
Axiom 2. *People who work hard are successful.*
Axiom 3. *Successful people are not foolish.*
Axiom 4. *Unhappy people are foolish.*

Example 1. Prove the following statement: *If Bill works hard, then he will be happy.*

Solution:	*Bill works hard.*	Hypothesis
∴	*Bill is successful.*	Substitution and Detachment in Axiom 2
∴	*Bill is not foolish.*	Substitution and Detachment in Axiom 3
∴	*Bill is happy.*	Substitution and Contrapositive in Axiom 4
∴	*If Bill works hard, he is happy.*	

Example 2. Prove the following statement: *If Bill is unhappy, then he is not a student.*

Solution 1:	*Bill is unhappy.*	Hypothesis
∴	*Bill is foolish.*	Substitution and Detachment in Axiom 4
∴	*Bill is not successful.*	Substitution and Contrapositive in Axiom 3
∴	*Bill does not work hard.*	Substitution and Contrapositive in Axiom 2
∴	*Bill is not a student.*	Substitution and Contrapositive in Axiom 1
∴	*If Bill is unhappy, he is not a student.*	
Solution 2:	*Bill is unhappy.*	Hypothesis
∴	*Bill does not work hard.*	Substitution and Contrapositive in Theorem 1
∴	*Bill is not a student.*	Substitution and Contrapositive in Axiom 1
∴	*If Bill is unhappy, he is not a student.*	
Solution 3:	*Either Bill is a student or Bill is not a student.*	
	Assume that Bill is a student.	
∴	*Bill works hard.*	Substitution and Detachment in Axiom 1
∴	*Bill is happy.*	Substitution and Detachment in Theorem 1
	But Bill is unhappy.	Hypothesis
∴	*Bill is a student* is false.	
∴	*Bill is not a student.*	Principle of Disjunction
∴	*If Bill is unhappy, he is not a student.*	

We note that once a theorem has been proved, then it may be used in the same manner as our axioms and regarded as a true statement in further discussion. This procedure avoids the necessity of returning to our axioms in every case and enables us to provide shorter proofs in later theorems.

EXERCISE 1.8

In each of the following, state whether the reasoning is valid or invalid. If valid, state the logical principle used, and, if invalid, state the reason.

1. Hypotheses: All right angles are equal.
   $\angle A$ and $\angle B$ are right angles.
   Conclusion: $\angle A = \angle B$.

2. Hypotheses: If snow is falling, then it is winter.
   Snow is falling.
   Conclusion: It is winter.

3. Hypotheses: If two triangles are congruent, then they are equal in area.
   Two triangles are equal in area.
   Conclusion: The triangles are congruent.

4. Hypotheses: $a = 6$ *or* $a = 10$.
   $a \neq 6$.
   Conclusion: $a = 10$.

5. Hypotheses: $AB = CD$ or $AB \parallel CD$.
   $AB \parallel CD$.
   Conclusion: $AB \neq CD$.

6. Hypotheses: If $x = 2$, then $2x + 1 = 5$.
   $2x + 1 \neq 5$.
   Conclusion: $x \neq 2$.

7. Hypotheses: If $5x + 2 = 3x + 6$, then $2x = 4$.
   If $2x = 4$, then $x = 2$.
   Conclusion: If $5x + 2 = 3x + 6$, then $x=2$.

8. Hypotheses: If John studies, then he is successful.
   If John is successful, then he is happy.
   Conclusion: If John is happy, then John studies.

9. Hypotheses: John is successful if he studies.
   John does not study.
   Conclusion: John is not successful.

10. Hypotheses: John is successful only if he studies.
    John does not study.
    Conclusion: John is not successful.

On the basis of the following set of axioms prove each of the theorems in questions (11) and (12).

Axiom 1. *Every line is a set of points.*

Axiom 2. *For each line l, there is a point not on l.*

Axiom 3. *There exist at least two points.*

Axiom 4. *For every pair of distinct points, there is one and only one line containing these points.*

11. There are at least two lines containing any point.

12. There are at least three lines.

1.9. Disproof

When a statement involves a quantifier, either stated or implied, it may sometimes be difficult to prove that the statement is true. If we doubt the truth of the statement, then we may attempt to disprove it. This may sometimes be just as difficult as trying to prove the truth of the statement but often it is a simpler task.

The statement *For all* $n \in I^+$, $n^2 + n + 41$ *is a prime integer* asserts that the statement is true for all positive integers n. To prove the statement false, we have to prove that the negation *For at least one positive integer*, $n^2 + n + 41$ *is not a prime integer* is a true statement. All we have to do is to find one positive integer n such that $n^2 + n + 41$ is not prime. This is known as disproof by counter-example.

In this example, when $n = 41$.

$$\begin{aligned} n^2 + n + 41 &= 41^2 + 41 + 41 \\ &= 41(41 + 1 + 1) \\ &= 41(43) \qquad \text{which is not a prime integer.} \end{aligned}$$

There are actually many positive integral values of n for which $n^2 + n + 41$ is not prime. The smallest such value is $n = 40$.

The statement *For some positive integer* x, $x + 5 < 2$ asserts that we can find at least one such positive integer. To prove the statement true, we only have to demonstrate the existence of one such integer. To disprove the statement, we must prove that the negation *For all positive integers* x, $x + 5 \nless 2$ is a true statement. Although this is simple to prove in this example, it will obviously be more difficult to prove than the first type in many instances.

If

$$x + 5 < 2\,,$$

then

$$x < -3\,.$$

Therefore,

$$x \notin I^+\,.$$

A conditional statement, such as *If two triangles are similar, then they are congruent,* is false only if the antecedent is true and the consequent is false. To prove the statement false, we must demonstrate that it is possible to have two similar triangles which are not congruent. Any example will suffice and so this is really equivalent to our disproof of statements beginning "For all . . .". In fact, the statement *For all* $n \in I^+$, $n^2 + n + 41$ *is a prime integer* is basically a conditional statement and could be rewritten as *If* $n \in I^+$, *then* $n^2 + n + 41$ *is a prime integer.*

To disprove the assertion *If two triangles are similar, then they are congruent,* we may proceed as follows.

Let

$$\triangle ABC \text{ have } AB = 1 \text{ in., } BC = 2 \text{ in., } AC = 2 \text{ in.}$$

and

$$\triangle DEF \text{ have } DE = 3 \text{ in., } EF = 6 \text{ in., } DF = 6 \text{ in.}$$

$$\therefore \quad \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}.$$

$$\therefore \quad \triangle ABC \,|||\, \triangle DEF\,.$$

But

$$AB \neq DE\,.$$

$$\therefore \quad \triangle ABC \not\equiv \triangle DEF\,.$$

Therefore the statement is false.

In general, to disprove any statement, we must prove that the negation of that statement is a true statement.

EXERCISE 1.9

Prove that the following statements are false.

1. If a triangle is right-angled, then it is isosceles.
2. For all $n \in I^+$, $n^2 + n + 11$ is a prime integer.
3. For some $n \in I^+$, $n(n + 1)$ is an odd integer.
4. If a quadrilateral is cyclic, then its opposite angles are equal.
5. All triangles have the property of not being isosceles.
6. If $x > 5$, then $x + 2 > 10$.
7. For all $x \in I^+$, $x < 5$ or $x > 5$.
8. For some real number x, $x^2 + 1 = 0$.
9. $\sqrt{2}$ is a rational number.

10. For all real numbers a and b, $|a| + |b| = |a + b|$.
11. For some $n \in I^+$, $n^2 + 5n + 4$ is a prime integer.
12. If a quadrilateral is not a square, then it is not a rectangle.

1.10. Symbolic Logic (Supplementary)

As in all mathematical studies, a judicious use of symbols becomes valuable in the examination of logic beyond the elementary level. By the use of symbols, an algebra of logic may be developed. However, in the elementary study of logic, it is more important to understand the ideas of logic than to become adept at manipulating symbols. It is quite simple to develop rules for "symbol pushing" while forgetting completely what the symbols actually mean. However, once we understand the elementary ideas of logical reasoning and wish to examine more complex statements, an algebra of logic does become useful.

If we understand the algebra of set theory, then we can also perform the algebra of elementary logic. The two systems are isomorphic although usually slightly different symbols are used. Logical reasoning may be interpreted in terms of set operations. Two statements p and q may be handled in the same way as two sets A and B. The commonly-used terminology in both systems is listed below with equivalent notations in the same row.

	Logic		Set Theory
$p \wedge q$	(p *and* q)	$A \cap B$	(A intersection B)
$p \vee q$	(p *or* q *or both*)	$A \cup B$	(A union B)
$\sim p$	(*not* p)	A'	(A complement)
$p \Rightarrow q$	(*if* p, *then* q)	$A \subseteq B$	(A is a subset of B)

If

$$p = 3 \text{ is a natural number,}$$
$$q = \pi \text{ is a rational number,}$$

then

$$p \wedge q = 3 \textit{ is a natural number and } \pi \textit{ is a rational number,}$$
$$q \vee p = 3 \textit{ is a natural number or } \pi \textit{ is a rational number,}$$
$$\sim p = 3 \textit{ is not a natural number.}$$

We know that the negation of (p *and* q) is (*not* p) *or* (*not* q). In symbols,

$$\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q \, .$$

Compare this with the algebra of sets where

$$(A \cap B)' = A' \cup B' \, .$$

Similarly, $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$, which may be compared with

$$(A \cup B)' = A' \cap B' \, .$$

If

$$p = \triangle ABC \equiv \triangle DEF, \qquad q = \triangle ABC \;|||\; \triangle DEF,$$

then

$$p \Rightarrow q = \textit{If } \triangle ABC \equiv \triangle DEF, \textit{ then } \triangle ABC \;|||\; \triangle DEF.$$

This sentence states that the set of all congruent triangles is a subset of the set of all similar triangles. If A is the set of all congruent triangles and B is the set of all similar triangles, then

$$A \subseteq B.$$

In this case, A is a proper subset of $B(A \subset B)$. However, if

$$p = \textit{In } \triangle CDE, \ \angle CDE = 90^\circ, \qquad q = (CE^2 = CD^2 + DE^2),$$

then

A = *The set of all right-angled triangles,*
B = *The set of all triangles in which the measure of the area of the square on one side is equal to the sum of the measures of the areas of the squares on the other two sides, then*

$$p \Rightarrow q \quad \textit{and} \quad q \Rightarrow p.$$

Hence,

$$p \Leftrightarrow q.$$

Also,

$$A \subseteq B \quad \textit{and} \quad B \subseteq A.$$

Hence,

$$A = B.$$

We have seen that the conditional statement $p \Rightarrow q$ is equivalent to the statement (*not* p) *or* q and that the negation of $p \Rightarrow q$ is p *and* (*not* q). Symbolically,

(1) $$(p \Rightarrow q) \Leftrightarrow \sim p \lor q,$$

(2) $$\sim(p \Rightarrow q) \Leftrightarrow p \land \sim q.$$

Statement (2) can be proved by using Statement (1) and the statement

$$\sim(p \lor q) \Leftrightarrow \sim p \land \sim q.$$

Example 1. Prove that

$$(p \Rightarrow q) \Leftrightarrow (\sim p \lor q).$$

Solution:

$$\begin{aligned} \sim(p \Rightarrow q) &\Leftrightarrow \sim(\sim p \lor q) \\ &\Leftrightarrow \sim(\sim p) \land \sim q \\ &\Leftrightarrow p \land \sim q. \end{aligned}$$

Example 2. Prove that

$$\sim (p \Leftrightarrow q) \Leftrightarrow (p \wedge \sim q) \vee (q \wedge \sim p)\,.$$

Solution:

$$\begin{aligned}\sim (p \Leftrightarrow q) &\Leftrightarrow \sim [(p \Rightarrow q) \wedge (q \Rightarrow p)]\\ &\Leftrightarrow \sim (p \Rightarrow q) \vee \sim (q \Rightarrow p)\\ &\Leftrightarrow \sim (\sim p \vee q) \vee \sim (\sim q \vee p)\\ &\Leftrightarrow (p \wedge \sim q) \vee (q \wedge \sim p)\,.\end{aligned}$$

To avoid the temptation to reduce the whole operation to mechanical manipulation of symbols, it is instructive to translate the various statements into English.

$$\sim (p \Leftrightarrow q) \Leftrightarrow (p \wedge \sim q) \vee (q \wedge \sim p)$$

states that it is false that p is equivalent to q if and only if p is true and q is false, or q is true and p is false. We see that the symbolism does help to shorten a lengthy compound statement, and, in such a situation, is of value. Mechanical manipulation is certainly useful once we clearly understand the meanings of the symbols.

Symbols used for quantifiers are

$\forall_x$ (for all x) the universal quantifier ,
$\exists_x$ (for some x) the existential quantifier .

In using quantifiers, we are converting an open sentence into a statement provided we state the domain of the variable. The statement "For all $x \in N$, $x + 2 > 1$" may be written

$\forall_x, x \in N\ (x + 2 > 1)$, and this is a true statement.

Similarly,

$\forall_x, x \in I\ (x + 2 > 1)$ is a false statement

and

$\exists_x, x \in I\ (x + 2 > 1)$ is a true statement.

If S_x is an open sentence and x is an element of a specified domain, then

$\forall_x(S_x)$ and $\exists_x(S_x)$ are statements.

The negation of $\forall_x(S_x)$ is the statement $\sim \forall_x(S_x)$ or the statement $\exists_x(\sim S_x)$. Hence,

$$\sim \forall_x(S_x) \Leftrightarrow \exists_x(\sim S_x).$$

Similarly,

$$\sim \exists_x(S_x) \Leftrightarrow \forall_x(\sim S_x).$$

EXERCISE 1.10

1. If p = *5 and 6 are consecutive integers,*
   q = *all right angles are equal,*
   $r = 4 + 5 = 3$,

   write each of the following statements symbolically.

   (a) If 5 and 6 are consecutive integers, then all right angles are equal.
   (b) All right angles are equal if $4 + 5 = 3$.
   (c) All right angles are equal and $4 + 5 = 3$.
   (d) 5 and 6 are consecutive integers or $4 + 5 = 3$.
   (e) If $4 + 5 = 3$ and all right angles are equal, then 5 and 6 are consecutive integers.
   (f) $4 + 5 \neq 3$ and all right angles are equal.
   (g) 5 and 6 are not consecutive integers if $4 + 5 \neq 3$.
   (h) If $4 + 5 = 3$, then all right angles are unequal or 5 and 6 are not consecutive integers.
   (i) All right angles are equal if and only if $4 + 5 \neq 3$ and 5 and 6 are consecutive integers.

2. Prove each of the following for statements p, q, r.

   (a) $[(p \Rightarrow q) \Rightarrow r] \Leftrightarrow (p \wedge \sim q) \vee r$.
   (b) $[\sim (p \vee q) \Rightarrow r] \Leftrightarrow p \vee q \vee r$.
   (c) $\sim [(p \wedge q) \wedge \sim (q \wedge r)] \Leftrightarrow (\sim p \vee \sim q) \vee (q \wedge r)$

3. Write the negation for each of the following.

   (a) $p \wedge \sim q$
   (b) $\sim p \vee (q \wedge r)$
   (c) $\forall_x, x \in N$ ($2x$ is even) in two forms
   (d) $p \Rightarrow q$ in two forms
   (e) $\exists_x, x \in Re$ $(2x^2 + 7 = 3)$ in two forms
   (f) $\forall_x (P_x \Rightarrow Q_x)$ in three forms
   (g) $\exists_x (S_x \Rightarrow P_x)$ in three forms

4. If P_x is *x is a prime integer,*
   Q_x is *x is an even integer,*
   R_x is *x is divisible by 2,*

   translate each of the following into English.

   (a) $\forall_x (Q_x \Rightarrow R_x)$
   (b) $\exists_x (P_x \wedge Q_x)$
   (c) $\sim \exists_x (P_x \wedge Q_x)$
   (d) $\forall_x (\sim P_x \vee R_x)$
   (e) $\forall_x (P_x \Rightarrow \sim R_x)$
   (f) $\exists_x (P_x \Leftrightarrow Q_x)$

Chapter Summary

A statement is a meaningful declarative sentence which is either true or false, but not both.

Logical Connectives

p and q	Conjunction
p or q	Disjunction
not p	Negation
if p then q	Conditional
p if and only if q	Biconditional

Statement	Negation
p	*not p*
p or q	(*not p*) *and* (*not q*)
p and q	(*not p*) *or* (*not q*)
not p	p
$p \Rightarrow q$	*p and* (*not q*)

Converse of $p \Rightarrow q$ is $q \Rightarrow p$.

Inverse of $p \Rightarrow q$ is $(not\ p) \Rightarrow (not\ q)$.

Contrapositive of $p \Rightarrow q$ is $(not\ q) \Rightarrow (not\ p)$.

Quantifiers

For all . . .

For some . . .

Principles of Proof

Detachment
Equivalence
Disjunction
Contrapositive
Syllogism
Substitution

Symbolic Logic (supplementary)

REVIEW EXERCISE 1

1. Write the negation of each of the following statements. Determine whether the statement or the negation is true.
   (a) 7 and 11 are consecutive integers.
   (b) The inverse of a true statement is always a true statement.
   (c) The converse of a true statement is always a true statement.

(d) The contrapositive of a true statement is always a true statement.
(e) Opposite angles of a cyclic quadrilateral are supplementary or $2 + 1 = 5$.
(f) Opposite angles of a cyclic quadrilateral are supplementary and $2 + 1 = 5$.
(g) If a triangle is right-angled, then the triangle is isosceles.
(h) If Lisbon is the capital city of Corsica, then Montreal is in Texas.
(i) If Lisbon is the capital city of Corsica, then Ottawa is the capital of Canada.
(j) For all $x \in Re$, $x^2 - 1 = (x - 1)(x + 1)$
(k) For some $x \in Re$, $2(x + 1) = 2x + 1$.
(l) For all $x \in Re$, $3x + 2 = 15$.

2. Write the inverse, converse, and contrapositive of each of the following statements.
(a) If a triangle is right-angled, then the triangle is isosceles.
(b) If $2x + 3 = 9$, then $x = 3$.
(c) Two triangles are equal in area if they are congruent.
(d) If two integers are odd, then their sum is even.
(e) If Winnipeg is the capital city of Ontario, then Montreal is in Texas.
(f) Winnipeg is the capital city of Ontario only if Ottawa is the capital of Canada.
(g) A sufficient condition for two triangles to be similar is that they be congruent.
(h) A necessary condition for two triangles to be similar is that they be congruent.
(i) A sufficient condition for two lines to be parallel is the equality of the alternate angles formed by a transversal.
(j) A necessary condition for a quadrilateral to be a parallelogram is the equality of the opposite angles.
(k) If the opposite angles of a quadrilateral are supplementary the vertices are concyclic.
(l) If a line is a tangent to a circle, then it is perpendicular to a radius.

3. Rewrite each of the following statements using the words "sufficient condition", "necessary condition", "only if".
(a) If two triangles are congruent, then they are equal in area.
(b) If two lines are parallel, then the corresponding angles formed by a transversal are equal.
(c) If it is raining, then the streets are wet.
(d) If John studies logic, then he will never be confused.
(e) A rectangle is a square if it has one pair of adjacent sides equal.
(f) A rhombus is a square if it has one angle equal to a right angle.

(g) $2x^2 + 1 < 9$ if $-2 < x < 2$.

(h) If $x \in I^+$, then $2x + 1$ is an odd integer.

(i) A real number is an irrational number if it can be represented by a non-recurring decimal.

(j) The product of two integers m and n is even if each of m and n is even.

(k) The roots of a quadratic equation $ax^2 + bx + c = 0$ are real if $b^2 \geq 4ac$.

(l) If a, b, and c are the lengths of the hypotenuse and the two other sides, respectively, of a triangle, then $a^2 = b^2 + c^2$.

4. In each of the following, state whether the reasoning is valid or invalid. If valid, state the logical principle used and, if invalid, state why.

(a) Hypotheses: Parallel lines are coplanar lines which do not intersect.
Coplanar lines which do not intersect have the same slope.
Conclusion: Parallel lines have the same slope.

(b) Hypotheses: John is either at home or at school.
John is not at school.
Conclusion: John is at home.

(c) Hypotheses: $AB \parallel CD$ or $AB = CD$.
$AB \parallel CD$.
Conclusion: $AB \neq CD$.

(d) Hypotheses: If two lines are parallel, then they are coplanar.
If two lines are coplanar, then they are not skew.
Conclusion: If two lines are parallel, then they are not skew.

(e) Hypotheses: If a number is even, then it is divisible by 2.
27 is not divisible by 2.
Conclusion: 27 is not even.

(f) Hypotheses: If it is raining, then the streets are wet.
It is not raining.
Conclusion: The streets are not wet.

(g) Hypotheses: All men are mortal.
Mr. Smith is a man.
Conclusion: Mr. Smith is mortal.

(h) Hypotheses: All men are mortal.
A dog is mortal.
Conclusion: A dog is a man.

(i) Hypotheses: $\angle A$ is an obtuse angle if any only if $90° < \angle A < 180°$.
$90° < \angle A < 180°$.
Conclusion: $\angle$A is an obtuse angle.

(j) Hypotheses: If two triangles are congruent, then they are similar.
If two triangles are congruent, then they are equal in area.
Conclusion: If two triangles are similar, then they are equal in area.

5. Using the axioms given, prove each of the theorems which follow.

Axiom 1. *All mathematicians are logical.*
Axiom 2. *Careful people are not foolish.*
Axiom 3. *Discontented people are foolish.*
Axiom 4. *Logical people are careful.*
Theorem 1. Mathematicians are contented.
Theorem 2. Foolish people are not logical.
Theorem 3. Careless people are not mathematicians.

6. Prove that the following statements are false.
   (a) For all $n \in I^+$, $n^2 - n + 87$ is a prime integer.
   (b) For all $n \in I^+$, $2n^2$ is an odd integer.
   (c) For some $n \in I^+$, $n^2 + 2n$ is a prime integer.

Supplementary

7. Determine the truth or falsity of the following statements. The domain of both x and y is I.
   (a) $\forall_x(\exists_y(x = y + 1))$
   (b) $\exists_y(\forall_x(x = y + 1))$
   (c) $\exists_y(\forall_x(x + y = x))$

8. Prove the following statements for p, q, r.
   (a) $[p \vee (q \wedge r)] \Leftrightarrow [(p \vee q) \wedge r)]$
   (b) $[p \wedge (q \vee r)] \Leftrightarrow [(p \wedge q) \vee (p \wedge r)]$

9. Write the following statement symbolically.

   For all positive ϵ, there exists a positive δ such that the absolute value of the difference of $f(x)$ and $f(y)$ is less than ϵ whenever the absolute value of the difference of x and y is less than δ.

Chapter 2

FUNCTION AS A MAPPING

2.1. Relations

"Chris *is the brother of* Peter."

"Three *is greater than* two."

Each of these sentences involves a *binary* relation, which associates *two* objects. (All the relations considered in this chapter are binary.) Consider the sentence

$$x > y, \qquad \text{where} \quad x, y \in I .$$

This sentence defines a relation. The values $x = 5, y = -2$ satisfy this inequality, and we say that the *ordered pair* $(5, -2)$ is a member of the solution set of the inequality. The pairs $(-2, -3)$, $(1, 0)$ are also members of the solution set, but $(-1, 1)$ and $(3, 5)$ are not members.

Consider the set

$$A = \{ (x, y) \mid x, y \in I, \quad x > y \} .$$

We see that any ordered pair belonging to A satisfies the defining sentence of the relation. Conversely, any ordered pair that satisfies the defining sentence of the relation belongs to A. Thus, we may consider the relation to be the set A itself. In general, we call any set of ordered pairs a relation.

The set of all first components, x, of the ordered pairs in a relation is called the *domain* of the relation. The set of all second components, y, of the ordered pairs in a relation is called the *range* of the relation. For the relation above whose defining sentence is

$$x > y, \qquad x, y \in I ,$$

the domain is the set of integers, I, and the range is also I.

We may interpret *a relation as a subset of a Cartesian product.* Consider the Cartesian product $A \times B$ ("A cross B") where $A = \{1, 2, 3\}$ and $B = \{7, 8, 9\}$. Recall that $A \times B$ is the set of all ordered pairs (x, y) such that $x \in A$ and $y \in B$. Thus, nine ordered pairs may be formed; that is,

$$A \times B = \{(1, 7), (1, 8), (1, 9), (2, 7), (2, 8), (2, 9), (3, 7), (3, 8), (3, 9)\}.$$

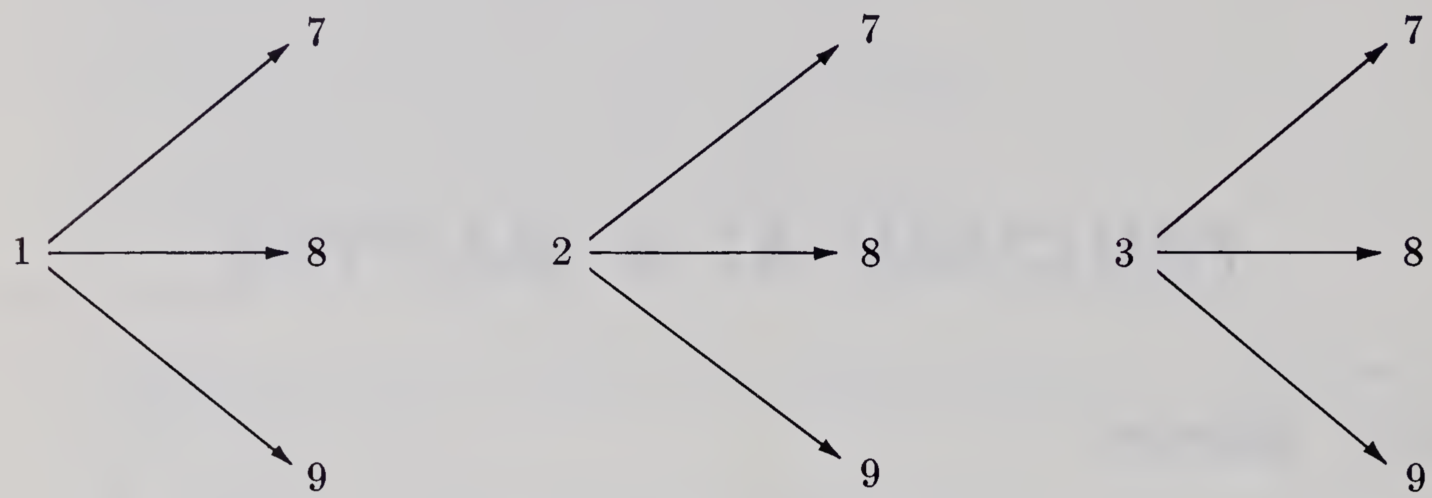


Figure 2.1

The relation whose defining sentence is

$$y = x + 5, \quad x \in A, \quad y \in B,$$

consists of the pairs (2, 7) and (3, 8). We may write

$$\{(x, y) \mid y = x + 5, x \in A, y \in B\} = \{(2, 7), (3, 8)\}.$$

Naming the set $\{(2, 7), (3, 8)\}$ as S, we say that S is a relation in $A \times B$.

Example 1.

(a) If $M = \{1, 2, 3, 4, 5\}$, list the members of the relation in $M \times M$ defined by

$$x^2 + y^2 < 25.$$

(b) List the members of (i) the domain, (ii) the range, of this relation.

Solution:

(a) By trial, we find the relation is described by the set P where

$$P = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (4,1), (4,2)\}.$$

(b) (i) The domain $D_P = \{1, 2, 3, 4\}$.
(ii) The range $R_P = \{1, 2, 3, 4\}$.

Example 2. Low temperature readings on a certain January 4 are given by the following table.

Ottawa	−10°F
Toronto	0°F
Kingston	−5°F
London	5°F
Windsor	10°F

List the ordered pairs in the relation defined by the sentence "City x has a lower temperature than city y". What are the domain and range of the relation?

Solution: If the relation involves the set T, then

$T = \{$(Ottawa, Kingston), (Ottawa, Toronto), (Ottawa, London), (Ottawa, Windsor), (Kingston, Toronto), (Kingston, London), (Kingston, Windsor), (Toronto, London), (Toronto, Windsor), (London, Windsor)$\}$

$D_T = \{$Ottawa, Kingston, Toronto, London$\}$

$R_T = \{$Kingston, Toronto, London, Windsor$\}$.

EXERCISE 2.1

1. State the domain and range of the relations described by the following sets.
   (a) $\{(3, 2), (3, 1), (3, 0)\}$
   (b) $\{(4, 5), (5, 6), (6, 7), (7, 8)\}$
   (c) $(x, y) \mid x, y \in N \text{ and } x + y < 6\}$
   (d) $\{(x, y) \mid x, y \in I \text{ and } xy = 4\}$

2. State which of the ordered pairs listed belong to the relation indicated.
   (a) $\{(x, y) \mid x, y \in Re \text{ and } x^2 > y^3\}$; $(1, \frac{1}{2})$, $(-4, 2)$, $(1, -5)$, $(0, -1)$, $(\sqrt{5}, \pi)$
   (b) $\{(x, y) \mid x, y \in Re \text{ and } |y| > x + 2\}$; $(-3, 0)$, $(100, 98)$, $(\sqrt{2}, \sqrt{\pi})$, $(-\frac{1}{3}, -\frac{1}{5})$, $(\sqrt{61}, 5)$.

3. List the ordered pairs in $M \times N$ if
   $$M = \{-2, 0, 2\} \quad \text{and} \quad N = \{3, 5, 7, 9\}.$$

4. What is (a) the intersection, (b) the union of M and N in question (3)?

5. From $M \times N$ in question (3), select the ordered pairs (a, b) such that
   $$a = b - 5.$$
   If the relation thus defined is denoted by P, list the members of D_P and R_P.

6. What is the domain of the relations in $I \times I$ defined by the following sentences?
   (a) $y = \dfrac{5}{x}$
   (b) $y = \dfrac{x + 2}{x + 1}$
   (c) $y = \dfrac{3}{x^2 - 4}$
   (d) $y = \dfrac{5}{x^2 + 2}$

(e) $y = \sqrt{x}$ (f) $y = |x|$

(g) $y = \dfrac{x}{x^2 - x - 6}$ (h) $y = 2^x$

(i) $y = -x^2$ (j) $y^2 = -x$

7. List four members for each of the following infinite sets. In each case, the relation is a subset of $Re \times Re$.

(a) $\{(a, b) \mid ab < 4\}$ (b) $\{(r, s) \mid s = 3r + 1\}$

(c) $\{(p, q) \mid q = 0\}$ (d) $\{(x, y) \mid x < 4 \text{ and } y = x\}$

(e) $\{(a, b) \mid 3 < a < 5\} \cap \{(a, b) \mid b = a^2\}$

(f) $\{(x, y) \mid y = |x|\} \cap \{(x, y) \mid |x| \leq 2\}$

2.2. Graphs of Relations

If both the domain and range of a relation are subsets of Re, the graph of the relation is the set of points whose co-ordinates are the ordered pairs in the relation.

Example 1. Draw the graph of the relation whose defining sentence is

$$x^2 + y^2 = 25, \quad x, y \in I.$$

Solution: The pairs of integers that satisfy this equation are $(0, 5)$, $(0, -5)$, $(5, 0)$, $(-5, 0)$, $(3, 4)$, $(-3, 4)$, $(3, -4)$, $(-3, -4)$, $(4, 3)$, $(-4, 3)$, $(4, -3)$, and $(-4, -3)$; the graph is shown in Figure 2.2.

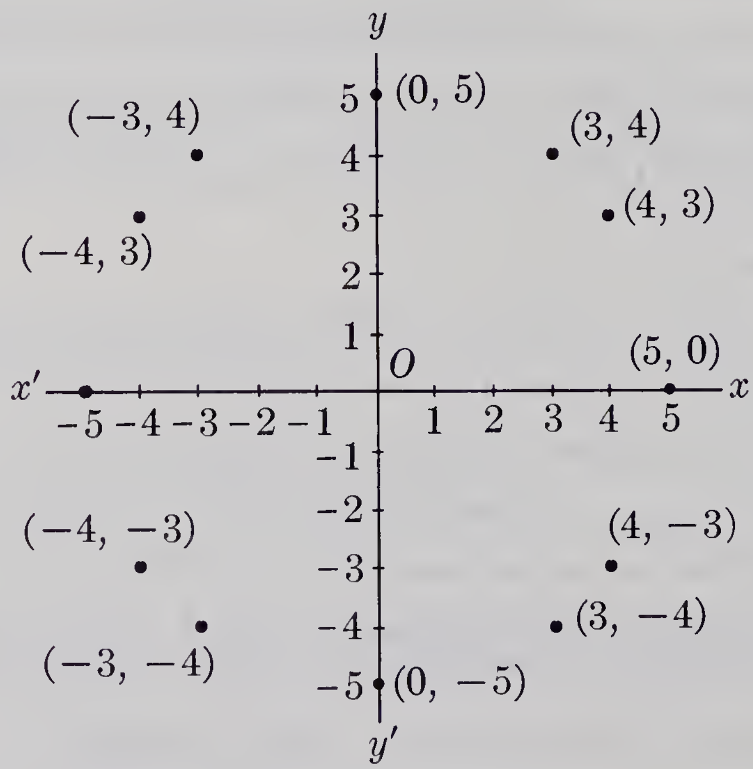


Figure 2.2

Example 2. Draw the graphs of the relations in $Re \times Re$ defined by

(i) $x^2 + y^2 = 25$, (ii) $x^2 + y^2 < 25$, (iii) $x^2 + y^2 \geq 25$.

Solution:

(i) Referring to our solution for Example 1, we see that the graph of $x^2 + y^2 = 25$ includes all those points shown in Figure 2.3(i) and is a circle with centre at the origin and radius 5.

(ii) The graph of $x^2 + y^2 < 25$ contains all those points that lie inside the circle in (i) above. The required region is shown shaded in Figure 2.3(ii). The circle itself is shown as a broken line to indicate that it is not part of the required graph.

(iii) The graph of $x^2 + y^2 \geq 25$ consists of the points on the circle in (i) and the points outside the circle. The diagram shows only part of the graph.

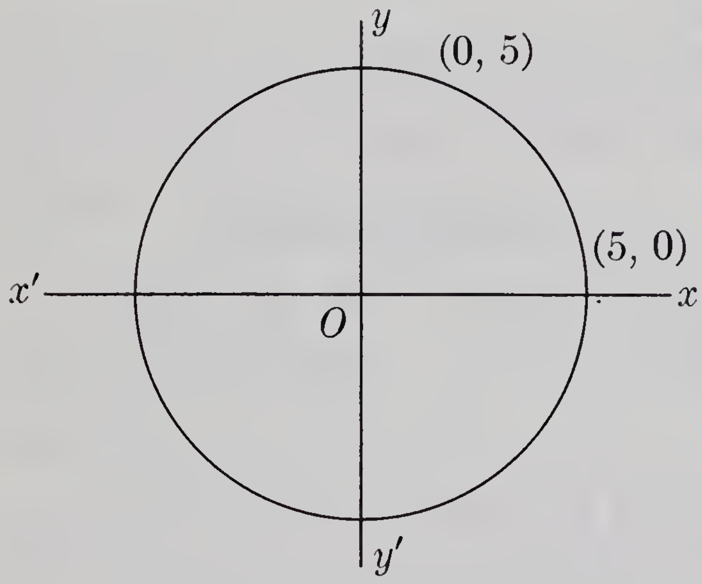


Figure 2.3 (i)

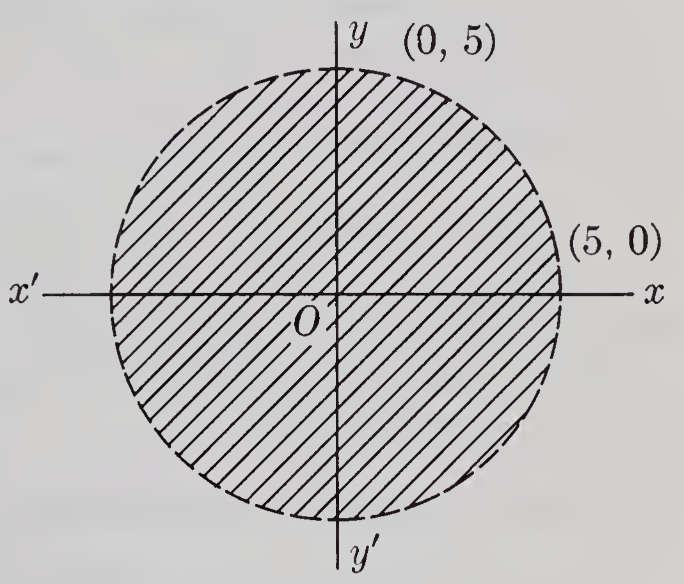


Figure 2.3 (ii)

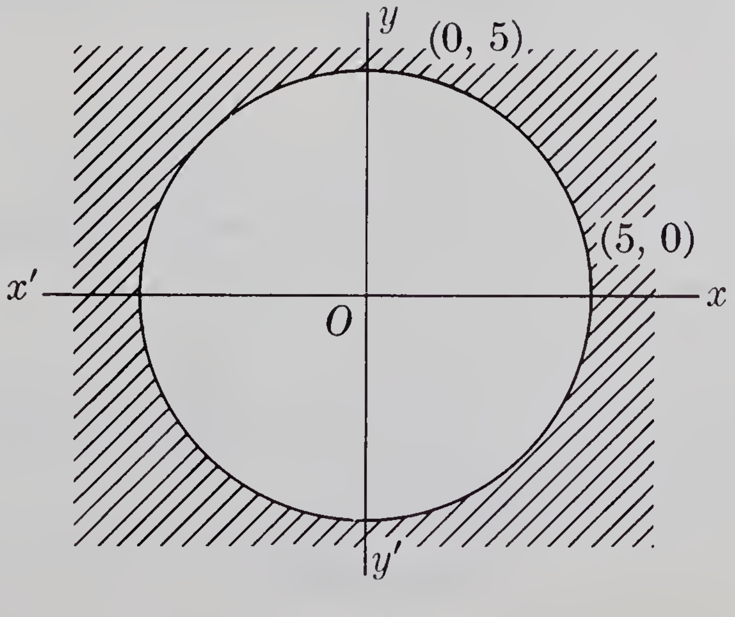


Figure 2.3 (iii)

EXERCISE 2.2

1. State the condition on the co-ordinates of a point (x, y) for the point to lie in the (a) first, (b) second, (c) third, (d) fourth quadrant.

2. State, if possible, the x- and y-intercepts of the graphs defined by the following sentences. $(x, y \in Re)$

   (a) $x + y = 1$ (b) $2x + y = 5$ (c) $y = x^2$
   (d) $y - 1 = 3(x - 2)^2$ (e) $x^2 + y^2 = 30$ (f) $xy = 0$.

3. Describe in words the graphs of each of the following relations in $Re \times Re$.

   (a) $\{(x, y) \mid y = 4\}$ (b) $\{(x, y) \mid x = -2\}$
   (c) $\{(x, y) \mid y = 0\}$ (d) $\{(x, y) \mid x = 0\}$
   (e) $\{(x, y) \mid y > 0\}$ (f) $\{(x, y) \mid x > 0\}$
   (g) $\{(x, y) \mid y > 0 \text{ and } x > 0\}$ (h) $\{(x, y) \mid y < 0 \text{ and } x < 0\}$

4. Given that the domain for each relation is $\{-4, -2, 0, 2, 4\}$, list the ordered pairs and draw the graph of the following.

   (a) $\{(a, b) \mid b = a^2\}$ (b) $\{(x, y) \mid y = |x|\}$
   (c) $\{(m, n) \mid n = 2m + 1\}$ (d) $\{(s, t) \mid t = -s\}$

5. Graph the relations in $Re \times Re$ defined by the following sentences.

   (a) $5x + 3y = 15$ (b) $y = x^2$ (c) $x - y = 2$ (d) $x - y < 2$
   (e) $x - y > 2$ (f) $y = |x|$ (g) $y > |x|$ (h) $y < |x|$
   (i) $|x - y| < 2$

6. Draw the graphs of the following relations in $Re \times Re$.

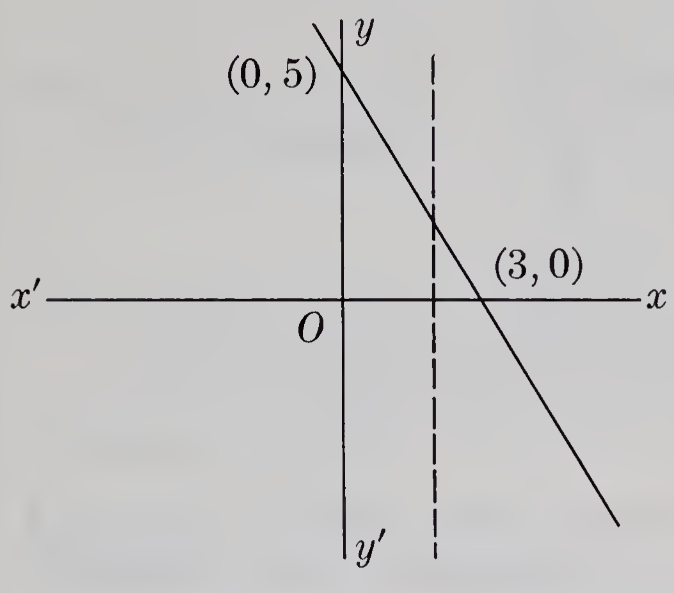
   (a) $\{(x, y) \mid 3x - 2y > 6\} \cap \{(x, y) \mid 3x - 2y < 12\}$
   (b) $\{(x, y) \mid x^2 + y^2 \leq 25\} \cap \{(x, y) \mid y \geq x\}$
   (c) $\{(x, y) \mid -x^2 \leq y \leq x^2\}$
   (d) $\{(x, y) \mid (x - 1)^2 + (y + 2)^2 < 4\}$
   (e) $\{(x, y) \mid x^2 + 2x + y^2 - 2y \leq 23\}$
   (f) $\{(x, y) \mid x^2 + y^2 < 16\} \cap \{(x, y) \mid |x - y| < 2\}$
   (g) $\{(x, y) \mid |x - y| \leq 3\} \cap \{(x, y) \mid 3y > x\} \cap \{(x, y) \mid 3y < 2x\}$
   (h) $\{(x, y) \mid x^2 - 4nx + y^2 < 1 - 4n^2, n \in I\}$
   (i) $\{(x, y) \mid x^2 + y^2 = 1, y = nx, n \in I\}$
   (j) $\{(x, y) \mid x^2 + y^2 \leq 16\} \cap \{(x, y) \mid x^2 - 12x + y^2 + 11 \leq 0\}$

7. Graph in $I \times I$ the relation defined by

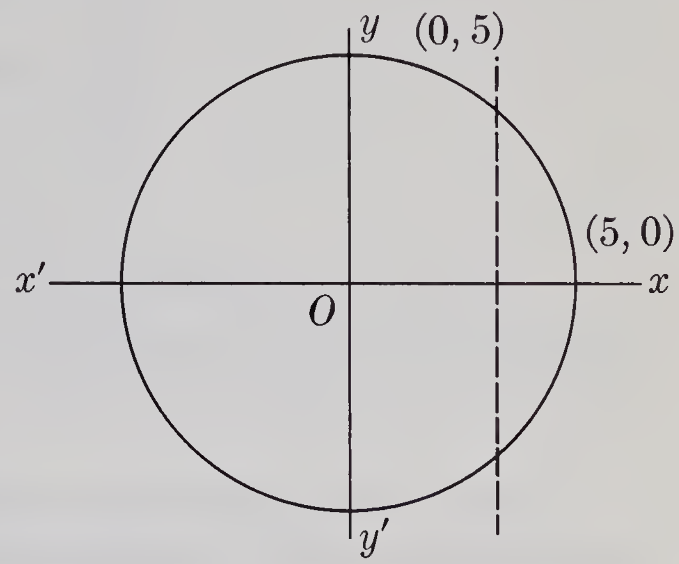
$$x^4 + y^4 \leq 256\,.$$

2.3. Function as a Mapping

Consider the graphs of the two relations shown in Figure 2.4



$5x + 3y = 15, \quad x, y \in Re$



$x^2 + y^2 = 25, \quad x, y \in Re$

Figure 2.4

Note that, in the ordered pairs that satisfy $5x + 3y = 15$, *for each value of x there is exactly one value of y. No two ordered pairs in the relation have the same first element. Such a relation is called a function. A line drawn parallel to the y-axis meets the graph in, at most, one point.*

Note that, in the ordered pairs that satisfy $x^2 + y^2 = 25$, some values of x are paired with more than one value of y. It is possible to find two ordered pairs in the relation that have the same first element. Such a relation is *not* a function. A line drawn parallel to the y-axis may intersect the graph in more than one point.

We may say that *a function is a relation in which each element in the domain is paired with one and only one element in the range.*

Consider the function whose defining sentence is

$$y = x + 2, \quad 0 \leq x \leq 3, \quad x \in I.$$

The graph of this function is shown in Figure 2.5.

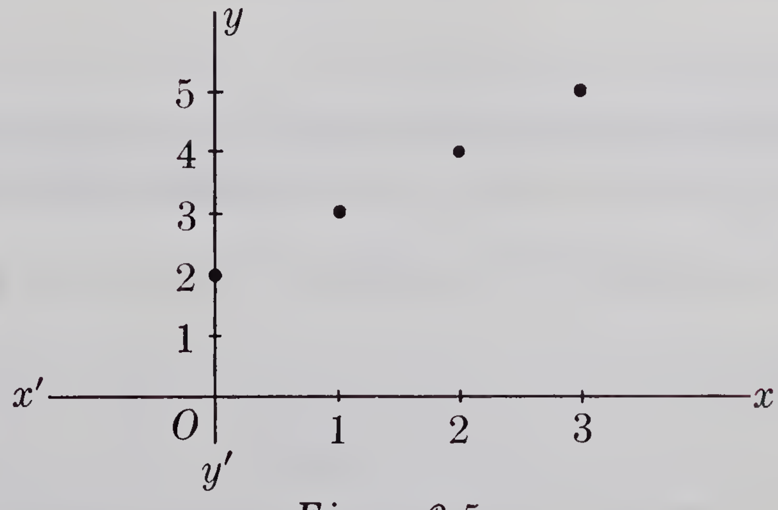


Figure 2.5

This same function may also be graphed in a different way by using two vertical number lines as in Figure 2.6.

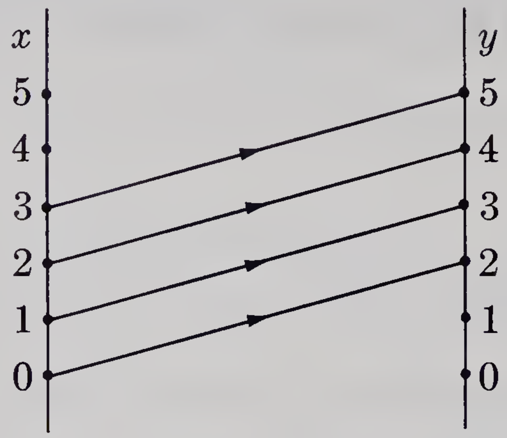


Figure 2.6

Figure 2.6 suggests that a function is a *mapping*. Each value of x is carried to or mapped onto its corresponding value of y. We may say that the function maps the elements of the domain onto the elements of the range. The function f graphed in Figure 2.6 is written as

$$f : x \to x + 2\,,$$

which is read as "the function f maps the number x onto the number $x + 2$." Any element $x + 2$ of the range is called the *image* of the corresponding element x of the domain. The symbol used to represent the image of x is $f(x)$, read as "f at x" or "f of x". In the example we have been discussing, $f(x) = x + 2$. Sometimes we write

$$f : x \to f(x)$$

to stress the fact that x is mapped onto its image $f(x)$ by the function f. Note that each value of x in the domain and its image $f(x)$ constitute an ordered pair $(x, f(x))$. The set of all such ordered pairs involved in the function f is identical with the set of ordered pairs obtained from our earlier definition of function as a special type of relation.

Sometimes we use the notation

$$f : A \to B$$

to stress the fact that a function maps the members of one set into those of another. In this notation, the function we have been discussing would be written in the form

$$f : \{0, 1, 2, 3\} \to \{2, 3, 4, 5\}, \qquad f(x) = x + 2\,.$$

In the notation

$$f : x \to f(x)\,,$$

the domain of f may be stated explicitly; if it is not, it is simply $\{x \mid f(x) \text{ is defined}\}$.

For example, if the function f is defined by

$$f : x \to \frac{1}{x}, \qquad f \text{ is in } Re \times Re\,,$$

the domain of f is $Re - \{0\}$, that is, $\{x \mid x \in Re, x \neq 0\}$.

In the notation

$$f : A \to B\,,$$

the domain of f is A. (Occasionally, we call f a mapping from A to B.)

The reader should have no difficulty in distinguishing these two notations for a function; it is always clear whether the arrow is preceded and followed by sets or individual elements.

Example 1. For the function

$$k : x \to x^2, \qquad x \in Re\,,$$

find $k(2)$, $k(1)$, $k(0)$, $k(-1)$, $k(-2)$.

Solution: If

$$k(x) = x^2\,,$$

then

$$k(2) = 4\,,$$
$$k(1) = 1\,,$$
$$k(0) = 0\,,$$
$$k(-1) = 1\,,$$
$$k(-2) = 4\,.$$

Note that $k(x) = k(-x)$. The graph of the function k is symmetric with respect to the y-axis.

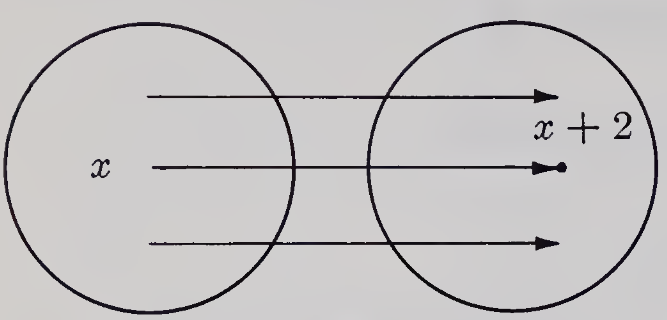
Note: The function

$$f : Re \to Re, \qquad f(x) = x + 2$$

maps the set Re *onto* the set Re; i.e., each element of the second set is the image of an element of the first set. Such a function is sometimes called *surjective.* (See Figure 2.7(i).) The function

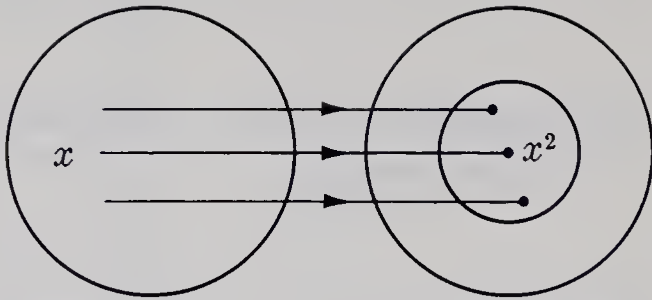
$$f : Re \to Re, \qquad f(x) = x^2,$$

as in Example 1 above, maps the set Re *into* Re but not *onto* Re; the set of image points is not Re but a proper subset of Re, namely, the set of nonnegative real numbers. (We may say that this function is not a surjective mapping from Re to Re.) (See Figure 2.7(ii) on following page.)



Re "onto" *Re*

(i)



Re "into" *Re*

(ii)

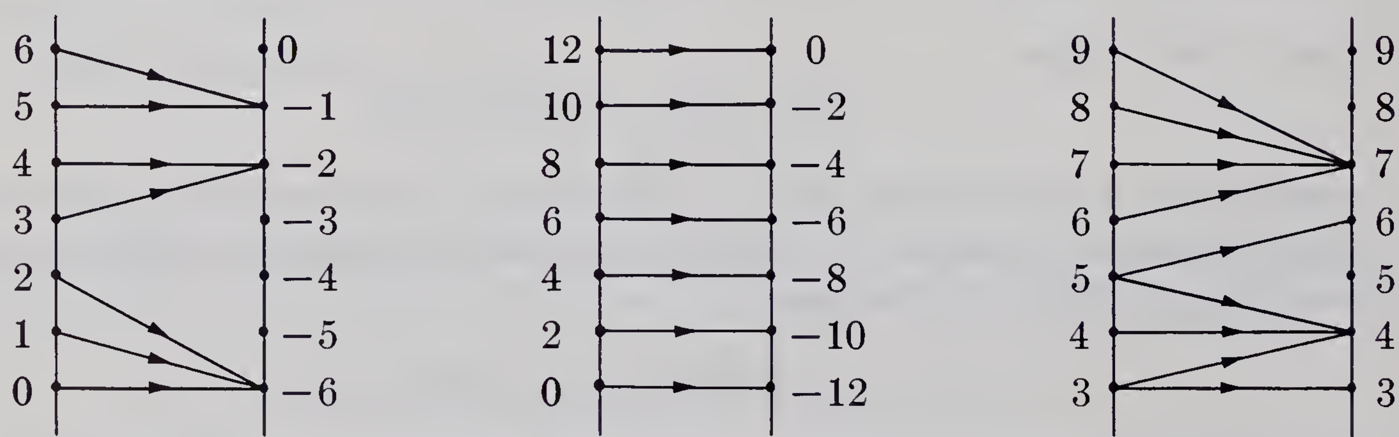
Figure 2.7

EXERCISE 2.3

1. If $g(x) = 2x - 5$, $\quad x \in Re$, state the value of each of the following.
   (a) $g(1)$ (b) $g(0)$ (c) $g(-1)$ (d) $g(\sqrt{2})$
   (e) $g(13)$ (f) $g(-40)$ (g) $g(2x)$ (h) $g(x + 1)$
   (i) $g[g(x)]$ (j) $g(k)$ (k) $g(x + a)$ (l) $g(\pi^2)$

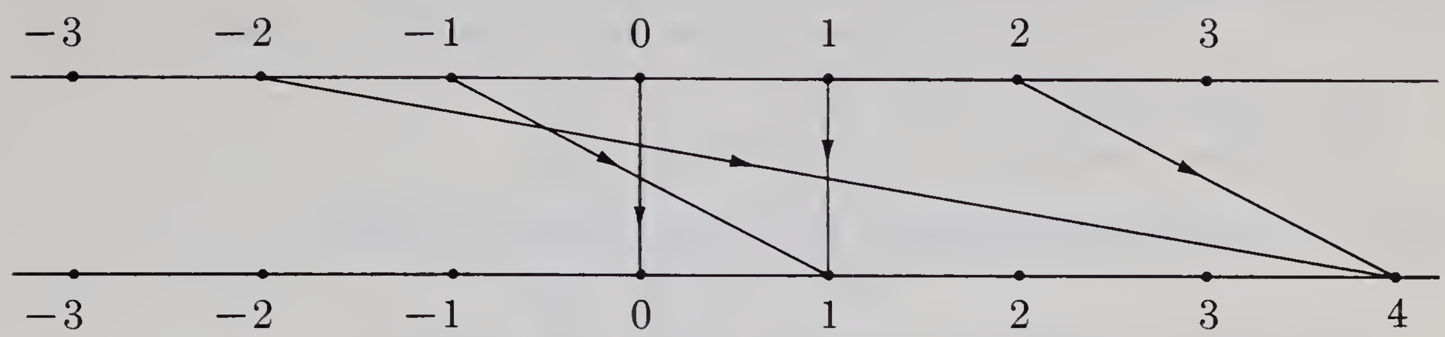
2. What is the image of -1 under each of the following mappings? $(x \in Re)$
   (a) $f : x \rightarrow -x$ (b) $g : x \rightarrow x$
   (c) $h : x \rightarrow 3x - 5$ (d) $m : x \rightarrow -x^3$
   (e) $s : x \rightarrow x^{2n}$ (f) $t : x \rightarrow (-x)^3$
   (g) $u : x \rightarrow (3x)^2$ (h) $v : x \rightarrow -\sqrt{x^2}$

3. Determine which of the following mappings are functions. List the sets of ordered pairs illustrated.

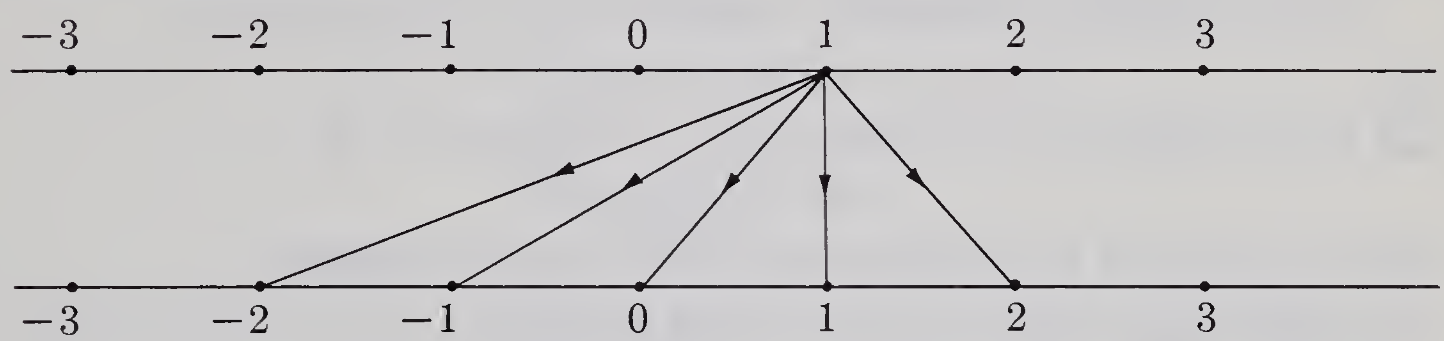


4. Which of the mappings shown below are functions? Explain.

(a)



(b)



5. Given that $-3 < x < 3$, $x \in I$, illustrate the following functions as mappings into the set of real numbers.

(a) $f : x \to 3x$ (b) $f : x \to 5 - x$

(c) $f : x \to \frac{3}{x}$, $x \neq 0$ (d) $f : x \to x^2 - 4$

6. Find the range for each of the mappings in question 5.

7. Given that the domain for each of the following mappings is $\{-3, -2, -1, 0\}$, graph the function using two vertical number lines in each case.

(a) $g : x \to 5x - 2$ (b) $h : x \to -x^2$

(c) $q : a \to (a + 1)^2$ (d) $r : b \to (2b - 1)^2$

8. The domain of a function g is $\{x \mid -2 \leq x \leq 3, x \in Re\}$. Find the range if

(a) $g(x) = x^3$, (b) $g(x) = |x|$.

9. For functions f and g, $f(x) = 7 - x$ and $g(x) = (x - 1)^2$. Find the members of $\{x \mid f(x) = g(x)\}$.

10. List five ordered pairs in each of the following functions and graph each function, using cartesian co-ordinates.

(a) $f = \{(x, y) \mid y = 3x^2 - 12x + 17, \quad x = 0, 1, 2, 3, 4\}$

(b) $g = \{(a, b) \mid ab = 8, \quad a \in Re^+\}$

(c) $h = \{(m, n) \mid n = 3 + |m + 1|, \quad m \in I\}$

(d) $k = \{(x, k(x)) \mid x - 2k(x) = 5, \quad x \in Re^+\}$.

11. f is a function such that, for all real numbers x and y,
$$f(x) + f(y) = f(x + y) .$$
(a) Prove that $f(0) = 0$. (Set $x = y = 0$.)
(b) Prove that $f(y) = -f(-y)$ for all $y \in Re$.

12. If s is a function such that for all real numbers t and u,
$$s(t - u) = s(t) \cdot s(u) ,$$
show that the range of s must be either $\{0\}$ or $\{1\}$.

13. List the mappings in question (2) that map the first set onto the second (that is, the surjective mappings). Consider Re to be the second set.

If
$$f : A \to B ,$$
and C is a subset of A, we define $f(C)$, the image of C, by
$$f(C) = \{f(c) \mid c \in C\} .$$
Find the images of the following sets under the given mappings.

14. $f : Re \to Re$, $f(x) = x + 1$, $\{x \mid 0 \leq x \leq 1\}$

15. $f : Re \to Re$, $f(x) = 2x^2$, $\{x \mid -1 \leq x \leq 3\}$

16. $f : Ra \to Ra$, $f(x) = -7x$, I

17. If $f : A \to B$, where $A \subset B$, and if $a \in A$ has the property that
$$f(a) = a ,$$
a is said to be a fixed point. What are the fixed points, if any, of the following functions?
(a) $f : Re \to Re$, $f(x) = x^2$
(b) $f : Re \to Re$, $f(x) = x + 1$
(c) $f : Re - \{1\} \to Re$, $f(x) = \dfrac{x}{1 - x}$
(d) $f : Re \to Re$, $f(x) =$ the least integer not less than x .

2.4. Inverse of a Relation

Consider the relation
$$A = \{(1, 4), (2, 5), (3, 6), (4, 7)\} .$$
Interchanging the components of each ordered pair, we obtain the relation
$$B = \{(4, 1), (5, 2), (6, 3), (7, 4)\} .$$
A and B are *inverse relations.* B is the inverse of A and the symbol used is A^{-1} ("inverse of A" or "A inverse"). We may also write $A = B^{-1}$, since A is the inverse of B.

DEFINITION. A^{-1}, the inverse of a relation A, is the relation obtained by interchanging the components of each ordered pair in A.

Figure 2.8 shows the graph of relation A and its inverse, A^{-1}.

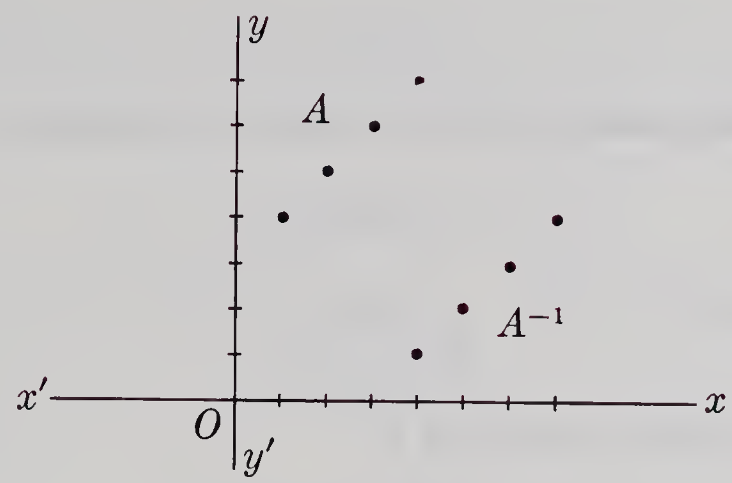


Figure 2.8

Note that the inverse of a relation is a relation. We have been discussing the function, which is a special type of relation. Consider the following mapping f and its inverse f^{-1}, which is also a function. The mapping $f: x \rightarrow 2x + 1$ is shown in Figure 2.9.

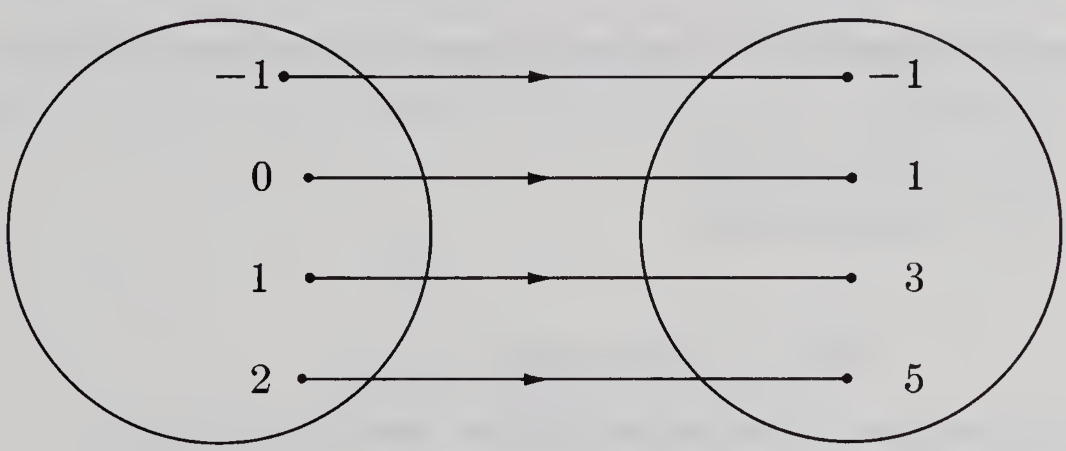


Figure 2.9

The ordered pairs in f are $(-1, -1)$, $(0, 1)$, $(1, 3)$, and $(2, 5)$. If the arrows are reversed as in Figure 2.10 the inverse of f is obtained.

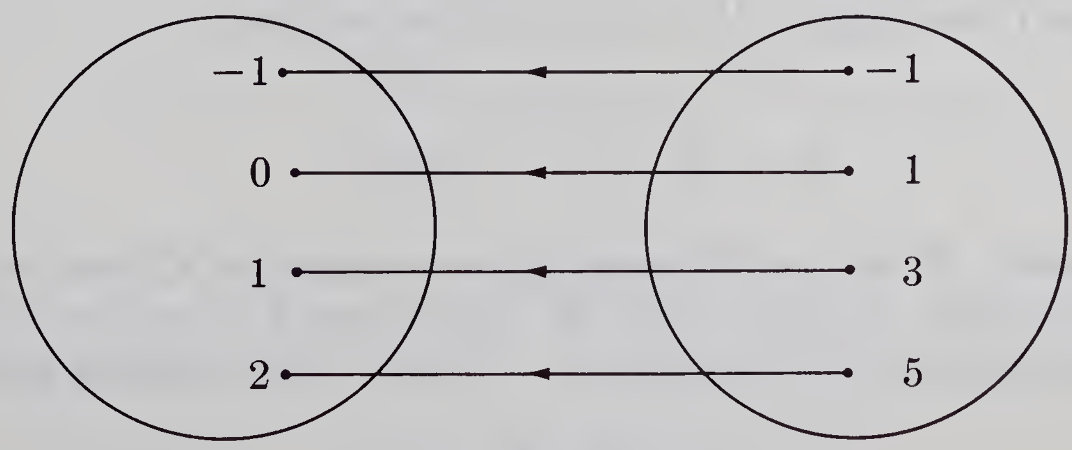


Figure 2.10

The inverse, f^{-1}, has as its ordered pairs

$$(-1, -1),\ (1, 0),\ (3, 1),\ (5, 2)\,.$$

If we write the defining sentence of f as

$$y = 2x + 1$$

then the defining sentence for f^{-1} is found by interchanging x and y.

$$x = 2y + 1$$

or

$$y = \frac{x - 1}{2}\,.$$

Using the notation which corresponds to

$$f : x \to 2x + 1\,,$$

we have

$$f^{-1} : x \to \frac{x - 1}{2}\,.$$

Is the inverse of a function always a function? Consider Example 1 below.

Example 1. List the ordered pairs in the inverses of each of the following functions. Is the inverse a function?

(a) $M = \{(2, 5),\ (3, 6),\ (4, 7)\}$

(b) $N = \{(4, 2),\ (5, 2),\ (6, 3)\}$.

Solution:

(a) $$M^{-1} = \{(5, 2),\ (6, 3),\ (7, 4)\}\,.$$

M^{-1} is a function because no two pairs have the same first element.

(b) $$N^{-1} = \{(2, 4),\ (2, 5),\ (3, 6)\}\,.$$

N^{-1} is not a function because two pairs have the same first element. In general, the inverse of a function is not always a function.

In Example 1, the relation M is defined by the sentence

$$y = x + 3\,.$$
$$M = \{(x, y) \mid y = x + 3\}\,.$$

The ordered pairs in M are such that each second component is found by *adding* 3 to the first component. In the inverse, M^{-1}, each second component is found by *subtracting* three from the first component. That is, if the defining sentence for M is

$$y = x + 3\,,$$

then the defining sentence for M^{-1} is

$$x = y + 3,$$

or

$$y = x - 3.$$

Hence, $M^{-1} = \{(x, y) \mid y = x - 3\}$.

Example 2. What is the defining sentence of the inverse of the function S, where

$$S = \{(x, y) \mid y = 5x - 4\}\ ?$$

Solution: Interchange the variables x and y. The defining sentence for S^{-1} is

$$x = 5y - 4$$

or

$$y = \frac{x + 4}{5}.$$

Hence,

$$S^{-1} = \left\{(x, y) \,\middle|\, y = \frac{x + 4}{5}\right\}.$$

Note that, if $(2, 6)$ is a member of a function then, $(6, 2)$ is a member of its inverse. We say that $(6, 2)$ is the *mirror image* of $(2, 6)$. We can show that the "mirror" is the line $y = x$; that is, show that the graph of the function and its inverse are symmetric with respect to the line $y = x$.

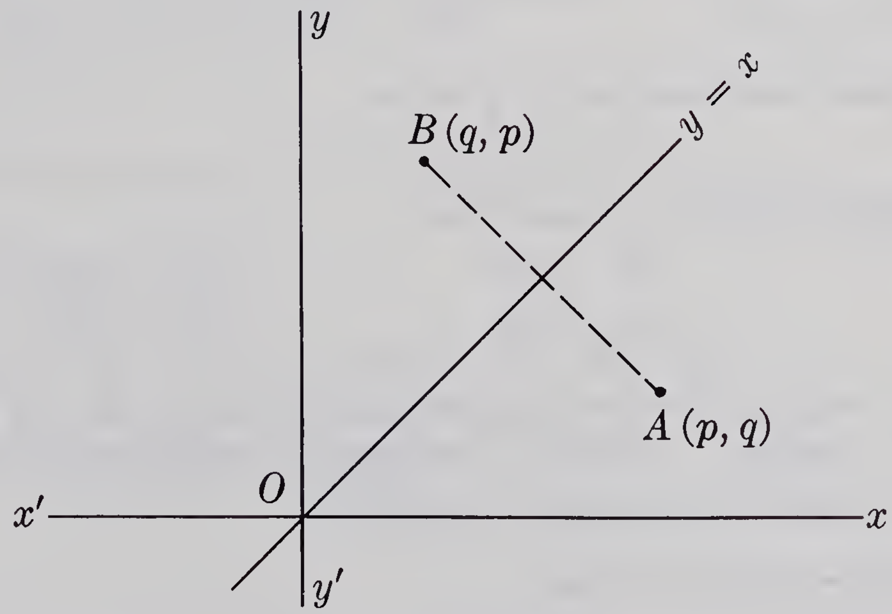


Figure 2.11

In Figure 2.11,

$$\text{Slope } AB = \frac{q - p}{p - q} = -1.$$

Since the slope of the line $y = x$ is 1, then AB is perpendicular to the line $y = x$. Also, the co-ordinates of the midpoint of AB are

$$\left(\frac{p+q}{2}, \frac{p+q}{2}\right)$$

and these co-ordinates satisfy the equation $y = x$. Hence, this midpoint lies on $y = x$ and $y = x$ is the perpendicular bisector of the line AB.

In general, the graph of the inverse of a relation is the mirror image in the line $y = x$, of the graph of the relation.

EXERCISE 2.4

1. List the ordered pairs in the mappings
   (i) $f: x \longrightarrow 5x + 2$
   (ii) $f: x \longrightarrow x^3$
   (iii) $f: x \longrightarrow |x|$
   if $x \in I$ and $|x| \leq 2$.

2. List the ordered pairs in the inverse of each mapping in question (1). Is the inverse a function in each case? Explain.

3. $$A = \{(4, 3), (5, 3), (6, 3)\}.$$
   List the pairs in A^{-1}. Is A^{-1} a function? Explain.

4. State the defining equation for the inverse of each of the following relations.
   (a) $\{(x, y) \mid 2y = 3x + 1\}$ (b) $\{(x, y) \mid y = x^3\}$
   (c) $\{(a, b) \mid b = a - 3\}$ (d) $\{(s, t) \mid s^2 - t^2 = 16\}$.

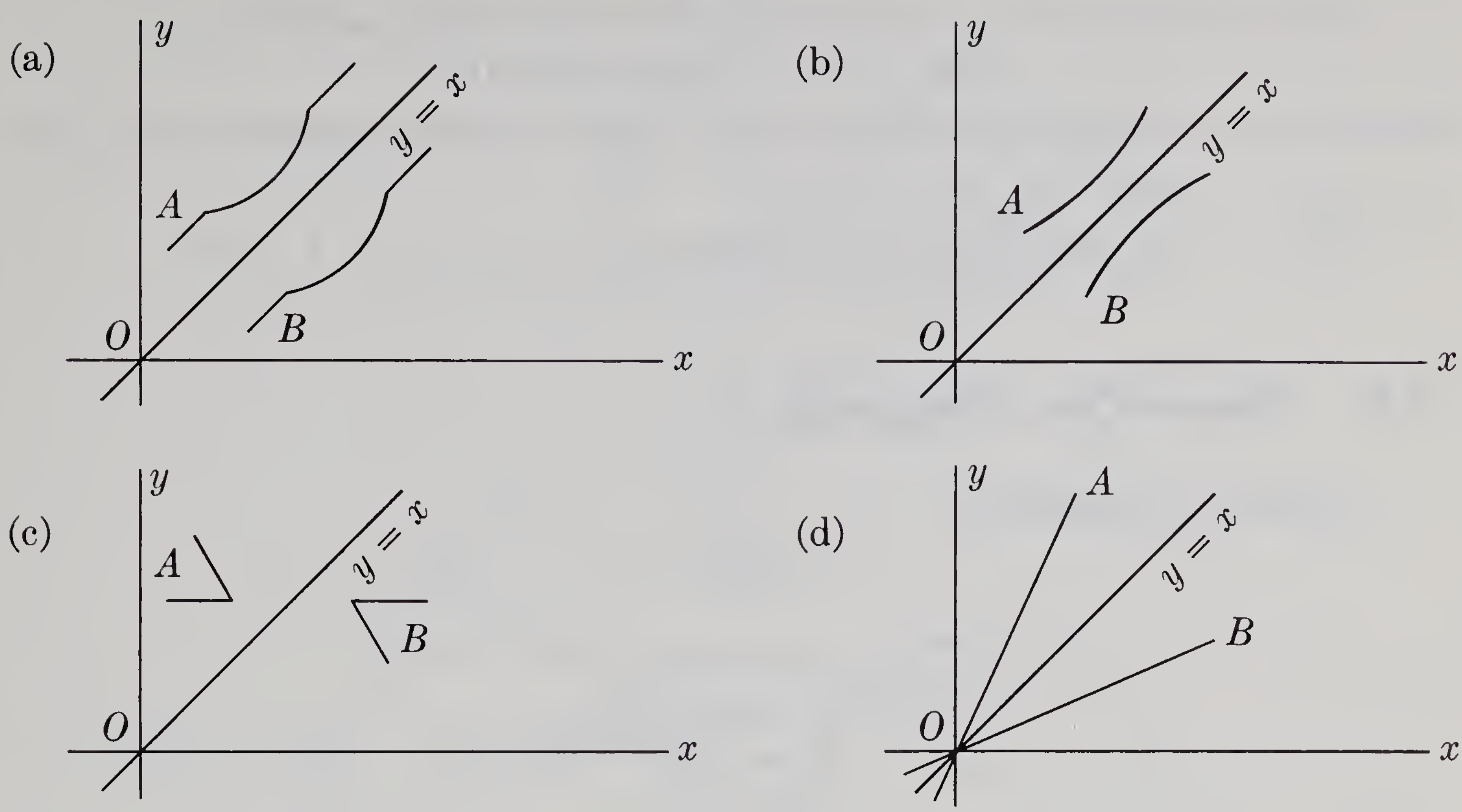
5. The following pairs are members of g where
   $$g = \{(x, y) \mid y = 2x - 5\}.$$
   State the missing component in each pair.
   (a) $(7, ?)$ (b) $(?, 9)$ (c) $(\frac{1}{2}, ?)$
   (d) $(?, 7)$ (e) $(?, 0)$ (f) $(0, ?)$

6. In question (5), what is the defining sentence for g^{-1}? Is g a function? Is g^{-1} a function? Explain. If each of the pairs in question (5) is considered as a member of g^{-1}, state the missing component.

7. How is the inverse of the function
   $$f: x \longrightarrow x$$
   related to f?

8. (a) For any function f, what is the inverse of f^{-1}?
   (b) What is the intersection of f and f^{-1} if
   $$f: x \longrightarrow x + 3?$$

9. In which of the cases illustrated below are A and B inverse functions?



10. Graph the function

$$f = \{(0,0),\ (1,1),\ (2,4),\ (3,9)\}$$

its inverse, f^{-1} and the line $y = x$. Is f^{-1} a function?

11. Graph the function

$$f = \{(x, y) \mid y = x^2,\ x \in Re,\ |x| \leq 3\}.$$

Graph the inverse of f. Is f^{-1} a function?

12. Repeat question (11) with

$$y = x^3$$

as the defining equation.

13. (a) Graph $f : x \longrightarrow \dfrac{3}{x+2}$
(b) Find the domain and range of f in $Re \times Re$.
(c) Find the domain and range of f^{-1}.
(d) Is f^{-1} a function? Explain.

14. If

$$f : A \longrightarrow B$$

and $b \in A$, we define $f^{-1}(b)$, the inverse image of b, by

$$f^{-1}(b) = \{a \mid a \in A \text{ and } f(a) = b\}.$$

Find the inverse images of the following points, under the given mappings.
(a) $f : Re \longrightarrow Re,\ f(x) = x^2;\ 1, 0, -3$
(b) $f : N \longrightarrow Re,\ f(x) = 3x + 4;\ 13,\ 2$

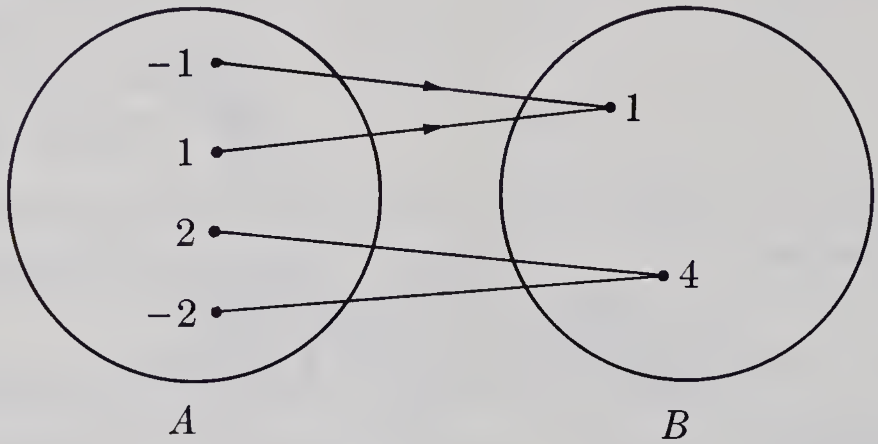
15. If
$$f : A \to B$$
and D is a subset of B, we define $f^{-1}(D)$, the inverse image of D, by
$$f^{-1}(D) = \{a \mid a \in A \text{ and } f(a) \in D\}.$$
Find the inverse images of the following sets, under the given mappings in $Re \times Re$

(a) $f : x \to x^2$, $\{x \mid 0 < x < 1\}$

(b) $f : x \to$ the least integer not less than x, $\{x \mid -1.5 < x \leq 3.7\}$.

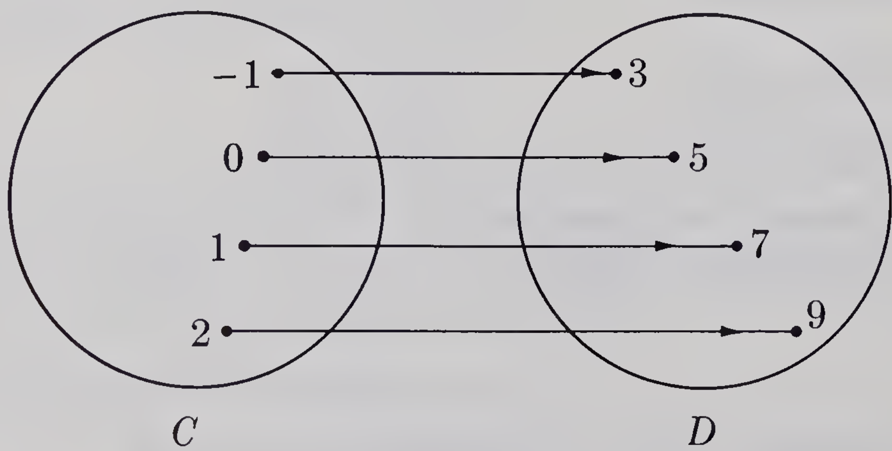
2.5. One-to-One Mappings

Consider the mapping
$$f : x \to x^2.$$



Note that, in the mapping f from domain A to range B, *two* elements of A are mapped onto *one* element of B.

Now consider the mapping
$$g : x \to 2x + 5.$$



If g is a mapping from the domain C to the range D, we note that each distinct element of C is mapped onto a separate and distinct element of D. Such a mapping or function is called *one-to-one*. (The mapping f is described as many-to-one.) Consider the one-to-one mapping g and its inverse, g^{-1}. Both are graphed in Figure 2.12.

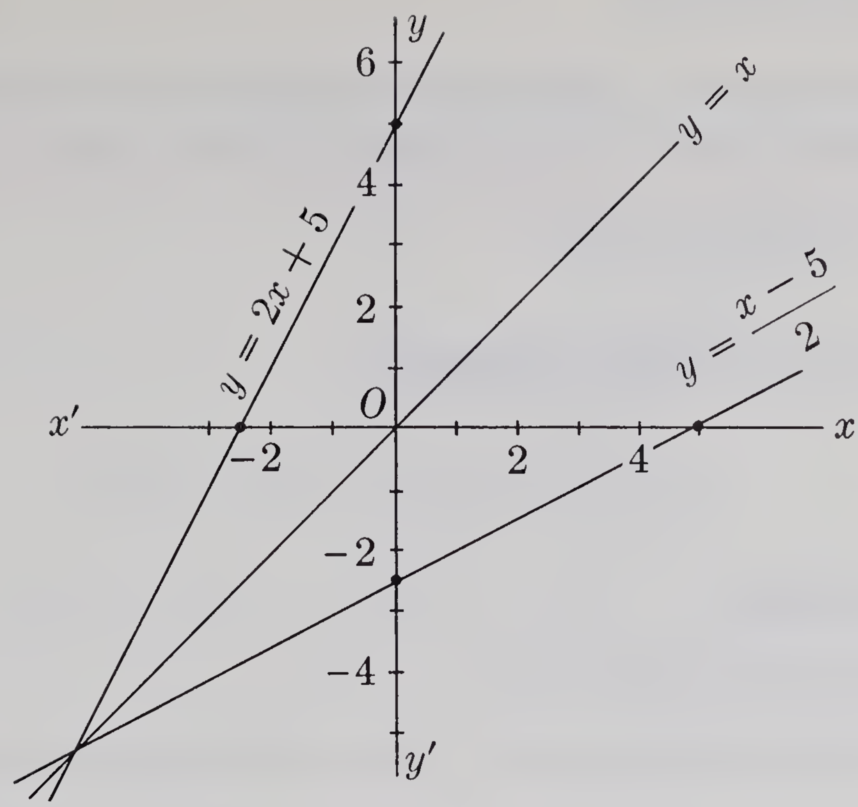


Figure 2.12

The inverse, g^{-1}, defined by

$$x = 2y + 5$$

or

$$y = \frac{x - 5}{2}$$

is also a function. The mapping

$$f : x \longrightarrow x^2$$

and its inverse f^{-1} are graphed in Figure 2.13. In this case, the inverse of f is not a function.

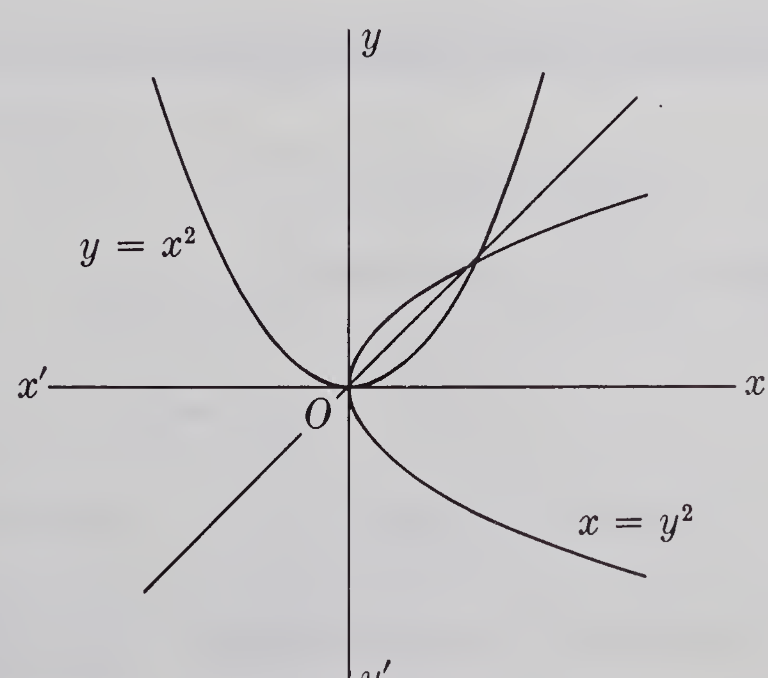


Figure 2.13

Note that the inverse of a many-to-one mapping is not a function. In general, although we shall not prove the fact, it can be shown that *the inverse of a function (or mapping) $f : A \to B$ is a function, if and only if, f is one-to-one.* In such a case, the inverse f^{-1} is also one-to-one.

Example 1. For the function f defined by

$$y = x^2 - 4x + 5, \qquad x \in Re, \quad x \geq 2,$$

find

(a) the range of f,

(b) the defining equation of f^{-1},

(c) the domain and range of f^{-1}. Also sketch the graph of f^{-1}.

Solution:

(a) If

$$y = x^2 - 4x + 5,$$
$$y - 1 = x^2 - 4x + 4,$$
$$y - 1 = (x - 2)^2.$$

If $x \geq 2$, the range for y is $\{y \mid y \geq 1\}$.

(b)

$$y = x^2 - 4x + 5$$

or

$$y - 1 = (x - 2)^2$$

is the defining equation for f, the defining equation for f^{-1} is

$$x - 1 = (y - 2)^2.$$

(c) Here, since $(y - 2)^2$ is nonnegative,

$$x - 1 \geq 0$$

and

$$x \geq 1.$$

The domain of f^{-1} is $\{x \mid x \geq 1\}$ and the range is $\{y \mid y > 2\}$.

Note: The domain of f is the range of f^{-1} and the range of f is the domain of f^{-1}.

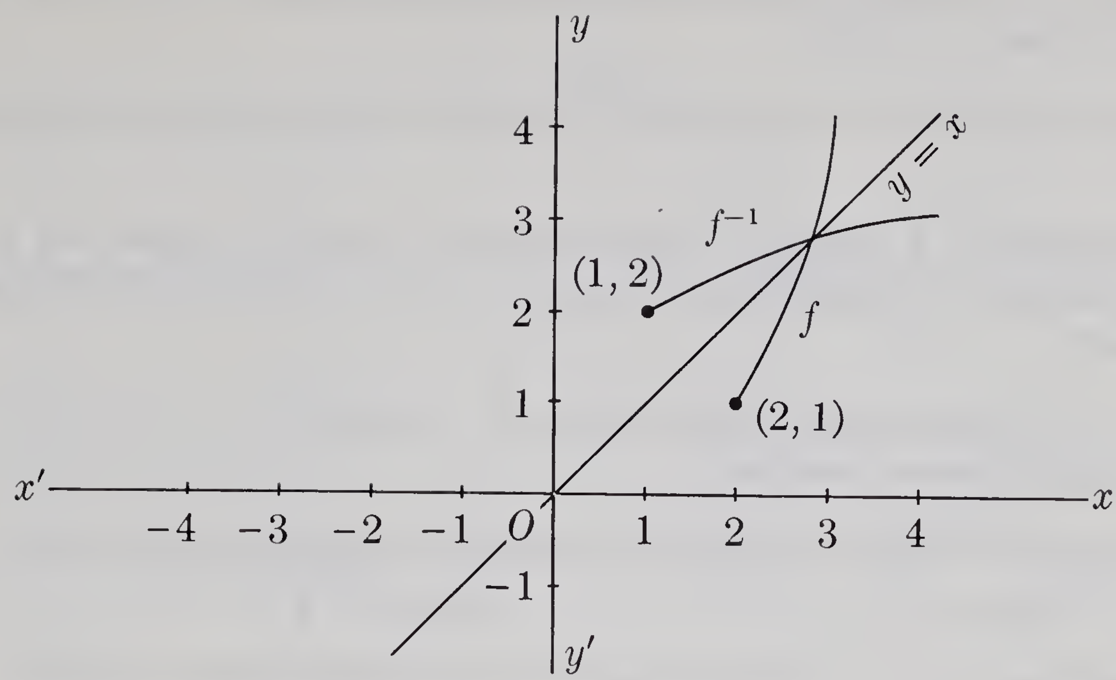


Figure 2.14

Note: A one-to-one mapping is said to be *injective.* A mapping that is both injective and surjective is said to be *bijective.*

EXERCISE 2.5

1. State the inverse of each of the following functions.
   (a) $\{(2, 6), (3, 4), (4, 1), (5, 0)\}$
   (b) $\{(0, 1), (4, 0)\}$
   (c) $\{(-4, 3), (4, 5), (-6, 0)\}$

If, for each of the following functions f, the image $f(x)$ is defined by the given equation, find (a) the range if the domain is as indicated, (b) an equation for $f^{-1}(x)$, (c) the domain and range of f^{-1}.

2. $f(x) = 5x - 4, \quad x \in Re$
3. $f(x) = \dfrac{3x - 5}{2}, \quad x \in Re$
4. $f(x) = \sqrt{x^2 - 16}, \quad x \in Re, x \geq 4$
5. $f(x) = \dfrac{1}{1 - x}, \quad x \in Re, x \neq 1$
6. $f(x) - x^2 - 6x + 9, \quad x \in Re, x \geq 3$
7. $f(x) = x^2 - 3, \quad x \in Re, x \geq \sqrt{3}$

Graph the following functions. In each case, state the range.

8. $f = \{(x, f(x)) \mid f(x) = -x, \quad x \leq 4\}$
9. $g = \{(x, g(x)) \mid g(x) = x^3, \quad x \leq 3\}$
10. $h = \{(x, h(x)) \mid h(x) = (x - 3)^2, \quad 0 \leq x \leq 6\}$

11. Graph the inverse of each function in questions (8) to (10). Is the inverse in each case a function?

12. For the mappings in questions (2) to (10), list those that are one-to-one (injective).

13. If the graph of a function is symmetric with respect to the line $y = x$, what relationship exists between the graph of the function and the graph of its inverse?

14. If f is a general linear function in Re defined by $y = ax + b$, show that f^{-1} is also a linear function.

15. Compare the relations defined by the following with their inverses $(x, y \in Re)$.
    (a) $x^2 + y^2 = 25$ (b) $xy = 8$.

16. In question (15) (a) and (b) what is the intersection of f and f^{-1}?

Chapter Summary

Cartesian product, ordered pair, relation, domain, range · Functions—Graphs of relations and functions · Function as a mapping · Inverse of a relation · Inverse of a function · One-to-one mappings

REVIEW EXERCISE 2

1. List the ordered pairs in $A \times B$ if

$$A = \{5, 4, 3\} \quad \text{and} \quad B = \{2, 7, 3, 1\}.$$

2. In question (1), what is (a) the intersection, (b) the union of A and B?

3. If $f(x) = |x| - 5$, state the value of
    (a) $f(2)$ (b) $f(-2)$ (c) $f(\frac{1}{2})$
    (d) $f[f(x)]$ (e) $f(x^2)$ (f) $f(a - b), a < b$

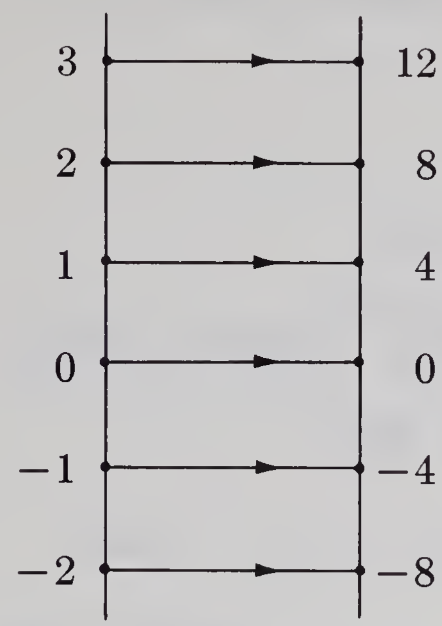
4. If $x \in Re$ find the domain and the range for each of the following functions.
    (a) $f: x \rightarrow x^2 - 2$ (b) $f: x \rightarrow \dfrac{8}{x}$
    (c) $f: x \rightarrow 2\sqrt{x}$ (d) $f: x \rightarrow 1 + |x|$
    (e) $f: x \rightarrow -|x|$ (f) $f: x \rightarrow \dfrac{1}{x^2 + 5}$

5. Use two horizontal number lines to draw the mappings in questions 4(a), 4(b), and 4(d), if $|x| \leq 4$ and $x \in I$.

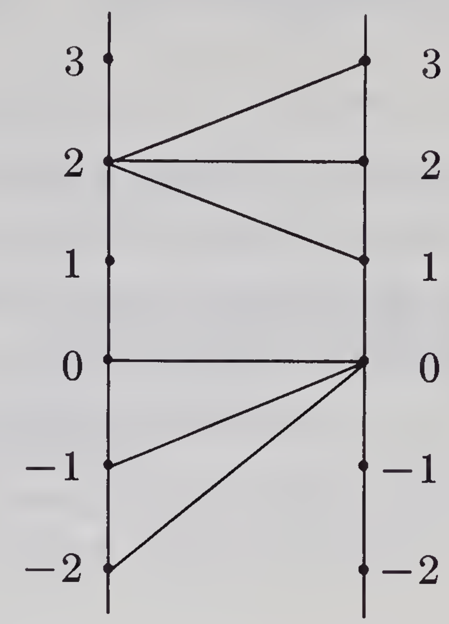
6. Use a Cartesian co-ordinate system to draw the graphs of the functions in 4(e) and 4(f) if $|x| \leq 3$.

7. Which of the following mappings is a function? Explain.

(a)



(b)



8. What is the image of 4 under each of the following mappings?

(a) $p : x \to \sqrt{x}$ (b) $q : x \to 3x - 10$

(c) $r : x \to \sqrt{x^2 + 9}$ (d) $s : x \to x^{2/3}$

(e) $t : x \to x^0$ (f) $u : x \to x^2 - 4x + 5$

9. List the pairs in B^{-1} if

$$B = \{(2, 3), (7, 4), (8, 5)\}$$

Is B^{-1} a function? Explain.

10. Graph the following functions over Re. In each case, state the range.

(a) $f = \{(x, y) \mid y = -2x, |x| \leq 3\}$

(b) $g = \{(x, y) \mid y = x^2 - 8x + 14, |x| \leq 7\}$

11. Graph the inverse of each function in question (10). Is the inverse in each case a function?

12. Draw the graphs of the following relations in $Re \times Re$.

(a) $\{(x, y) \mid 5x - 4y > 0\} \cap \{(x, y) \mid 5x - 4y < 20\}$

(b) $\{(x, y) \mid -(x - 2)^2 \leq (y - 5) \leq (x - 2)^2\}$

13. For the functions m and n, $m(x) = x^2 - 5x + 3$, and $n(x) = 51 - 3x$. Find the members of the set $\{x \mid m(x) = n(x)\}$.

14. For relation g, $g(x) = \sqrt{x}$. Find an expression for $g^{-1}(x)$. Find three points on the graph of g and the corresponding points on the graph of g^{-1}. Find the midpoint of the line joining each of the three pairs of points and show that the midpoint lies on the line $y = x$.

15. Sketch the graphs of g and g^{1-} in question (14). On a separate set of axes, sketch the graph of h where $h(x) = -\sqrt{x}$ and the graph of h^{-1}.

16. For the function (in $Re \times Re$)

$$t : x \to x^2 - 12x + 25,$$

find the domain and range. Also find the domain and range of t^{-1}.

17. For the mapping $f : x \to \dfrac{5}{x^2 + 5}$,
   (a) draw the Cartesian graph for $|x| \leq 5$
   (b) find the domain and range of f in $Re \times Re$,
   (c) on the same set of axes, graph f^{-1} for $|x| \leq 5$,
   (d) find the domain and range of f^{-1}.
   (e) Is f^{-1} a function? Explain.

18. Draw the graph of the region defined by

$$\{(x, y) \mid y > 3x^2, x \in Re\} \cap \{(x, y) \mid 2y < x + 5, \quad x \in Re\}.$$

19. If

$$f : A \to B,$$

C and D are subsets of A, and H and G are subsets of B, prove each of the following.
   (i) $f(C \cup D) = f(C) \cup f(D)$
   (ii) $f(C \cap D)$ is a subset of $f(C) \cap f(D)$.
   (iii) $f^{-1}(G \cup H) = f^{-1}(G) \cup f^{-1}(H)$
   (iv) $f^{-1}(G \cap H) = f^{-1}(G) \cap f^{-1}(H)$

Give an example of a function for which the inclusion of (ii) is proper.

20. In the notation of the previous question, prove that D is a subset of $f^{-1}(f(D))$ and that $f(f^{-1}(H))$ is a subset of H. Give examples of functions for which the inclusions are proper.

Chapter 3

SECOND DEGREE RELATIONS IN THE PLANE

3.1. The Circle

From our definition of a circle as a set of points in a plane that are a constant distance from a given point in the plane, we can, by using a rectangular co-ordinate system, develop an equation for the circle. Let $P(x, y)$ be any point in the plane.

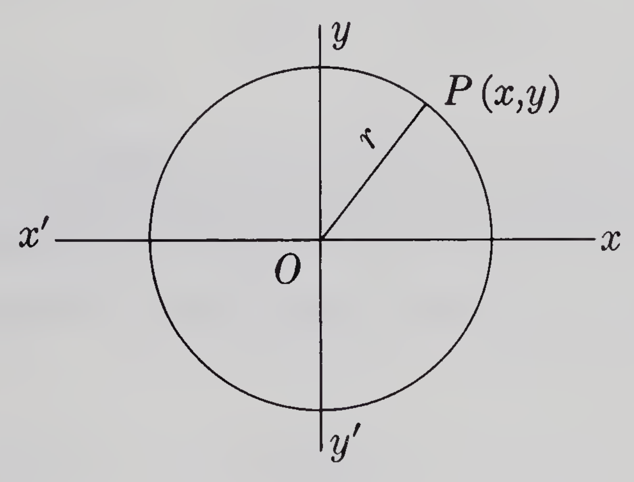


Figure 3.1

$P(x, y)$ lies on the circle with centre at the origin and radius r units if, and only if, $OP = r$. Therefore,

$$\sqrt{x^2 + y^2} = r \quad \text{or} \quad x^2 + y^2 = r^2.$$

Therefore, the equation of the circle is $x^2 + y^2 = r^2$.

Example 1. Find an equation of the circle with radius $3/2$ units and centre at the origin.

Solution: The equation of the circle is

$$x^2 + y^2 = \tfrac{9}{4},$$

or, with integral coefficients,

$$4x^2 + 4y^2 = 9\,.$$

Example 2. Find the radius of the circle represented by

$$25x^2 + 25y^2 = 8 .$$

Solution: An equation equivalent to the given equation is

$$x^2 + y^2 = \tfrac{8}{25} .$$

Therefore, the radius of the circle is $\sqrt{\tfrac{8}{25}}$ units or $\tfrac{2}{5}\sqrt{2}$ units.

We have previously studied the relation

$$C = \{(x, y) \mid x^2 + y^2 = 25, \quad x, y \in Re\} .$$

Let us sketch the graph of C by determining the intercepts, the domain, the range, and any symmetry that exists.

Intercepts

In the relation $x^2 = y^2 = 25$,

if

$$y = 0 ,$$

then

$$x^2 = 25 ,$$

and

$$x = \pm 5 .$$

and

if

$$x = 0 ,$$

then

$$y^2 = 25 ,$$

and

$$y = \pm 5 .$$

Therefore, the x-intercepts are ± 5. The points $(5, 0)$ and $(-5, 0)$ are on the graph. The y-intercepts are ± 5. The points $(0, 5)$ and $(0, -5)$ are on the graph.

We have established that the four points shown in Figure 3.2 are on the graph.

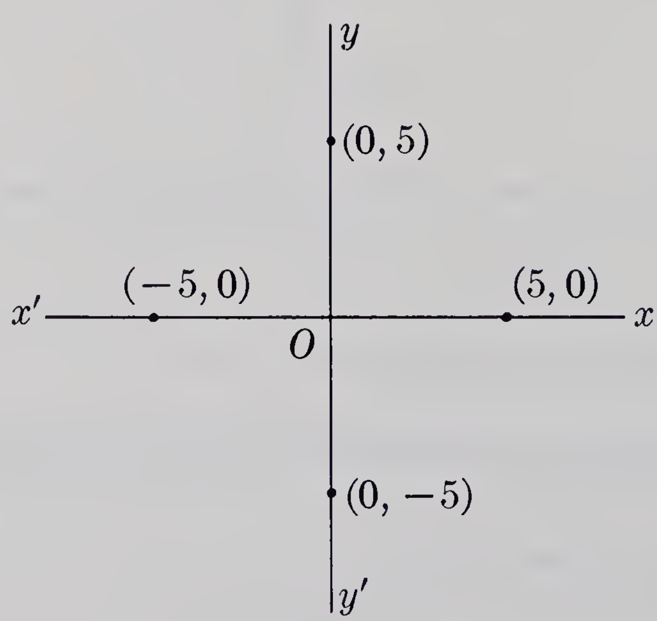


Figure 3.2

Domain

If

$$x^2 + y^2 = 25\,,$$

then

$$y^2 = 25 - x^2\,,$$

and

$$y = \pm\sqrt{25 - x^2}\,.$$

For $y \in Re$,

$$25 - x^2 \geq 0$$

and

$$x^2 \leq 25\,;$$

therefore,

$$|\,x\,| \leq 5\,.$$

The domain of the relation is $\{x \in Re \mid \ |\,x\,| \leq 5\}$.

Range

Similarly,

$$x^2 = 25 - y^2\,,$$

and

$$x = \pm\sqrt{25 - y^2}\,.$$

For $x \in Re$, $|\,y\,| \leq 5$. The range of the relation is $\{y \in Re \mid \ |\,y\,| \leq 5\}$.

We now know that the graph is confined to that region of the plane bounded by and inside the broken lines in Figure 3.3.

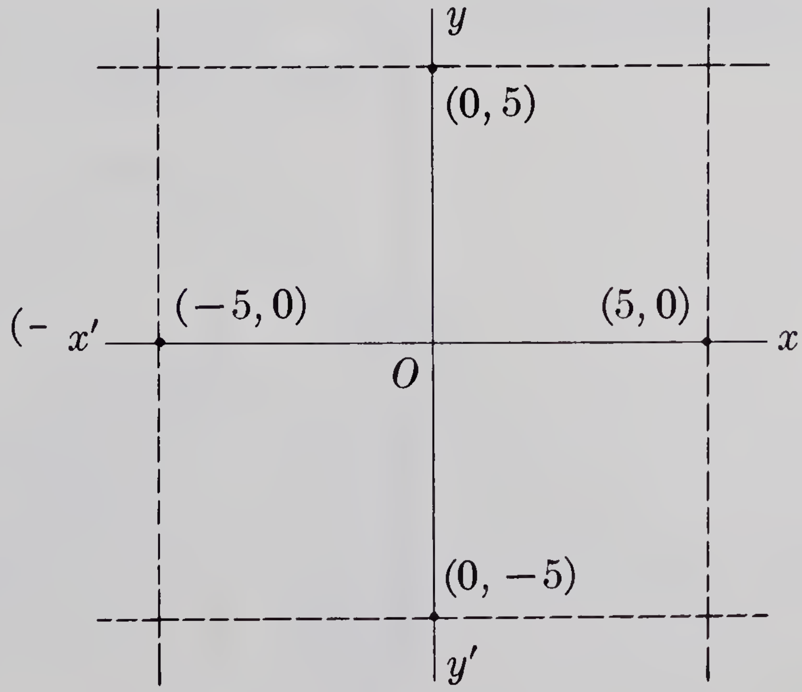


Figure 3.3

Symmetry

In the equation

$$x^2 + y^2 = 25\,,$$

if we replace x by $-x$ *or* if we replace y by $-y$, the equation is unchanged and so the graph is symmetrical with respect to the y-axis and the x-axis.

In the equation

$$x^2 + y^2 = 25\,,$$

if x is replaced by $-x$ *and* y by $-y$, the equation is unchanged and, therefore, the graph is symmetrical with respect to the origin.

Checking for symmetry enables us to sketch the graph quickly. For each point plotted in the first quadrant, we can immediately plot its *reflection* in the axis of symmetry or in the origin.

For $$x^2 + y^2 = 25\,,$$

a suitable table of values is

x	3	4
y	4	3

From the points $(3, 4)$ and $(4, 3)$, we can plot 6 other points by symmetry, as shown in Figure 3.4.

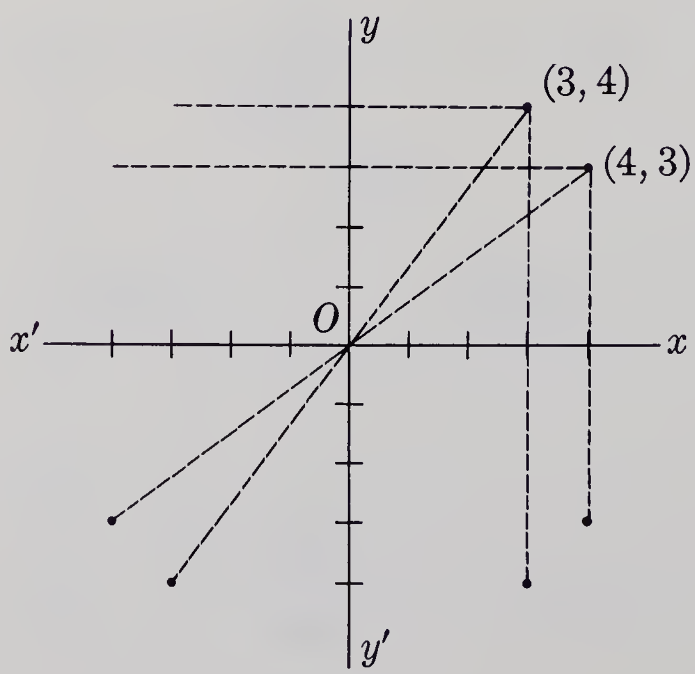


Figure 3.4

It appears that the complete graph of relation C is a circle with centre the origin and radius 5 as shown in Figure 3.5.

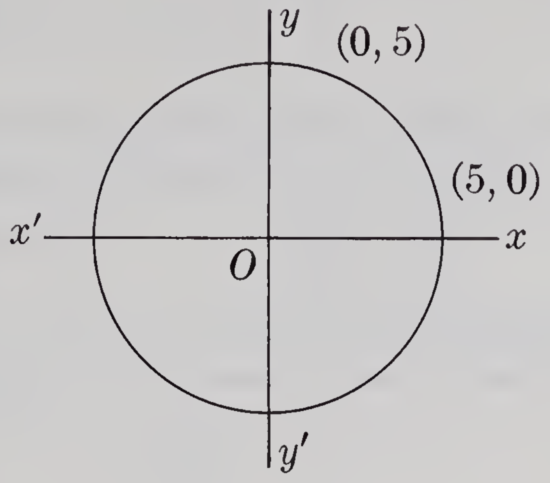


Figure 3.5

It should be noted that, if the graph of a relation is symmetrical about both the x-axis and the y-axis, it is symmetrical about the origin. The converse of this statement is not necessarily true. For example, the graph of $xy = 8$ is symmetrical about the origin but not symmetrical about either axis. Check by applying the three tests above. See question 5(h) in Exercise 3.1.

Example 3. Sketch the graph of

$$\{(x, y) \mid x^2 + y^2 < 25, \quad x > 0, \quad y < 0, \quad x, y \in Re\}.$$

Solution: The solution set of $\{(x, y) \mid x^2 + y^2 < 25, \quad x, y \in Re\}$ is the set of all points inside the circle whose equation is $x^2 + y^2 = 25$. The solution set of $\{(x, y) \mid x > 0, \quad y < 0, \quad x, y \in Re\}$ is the set of all points in the fourth quadrant not including the co-ordinate axes. The required solution set is the intersection of these two sets and is shown in Figure 3.5.

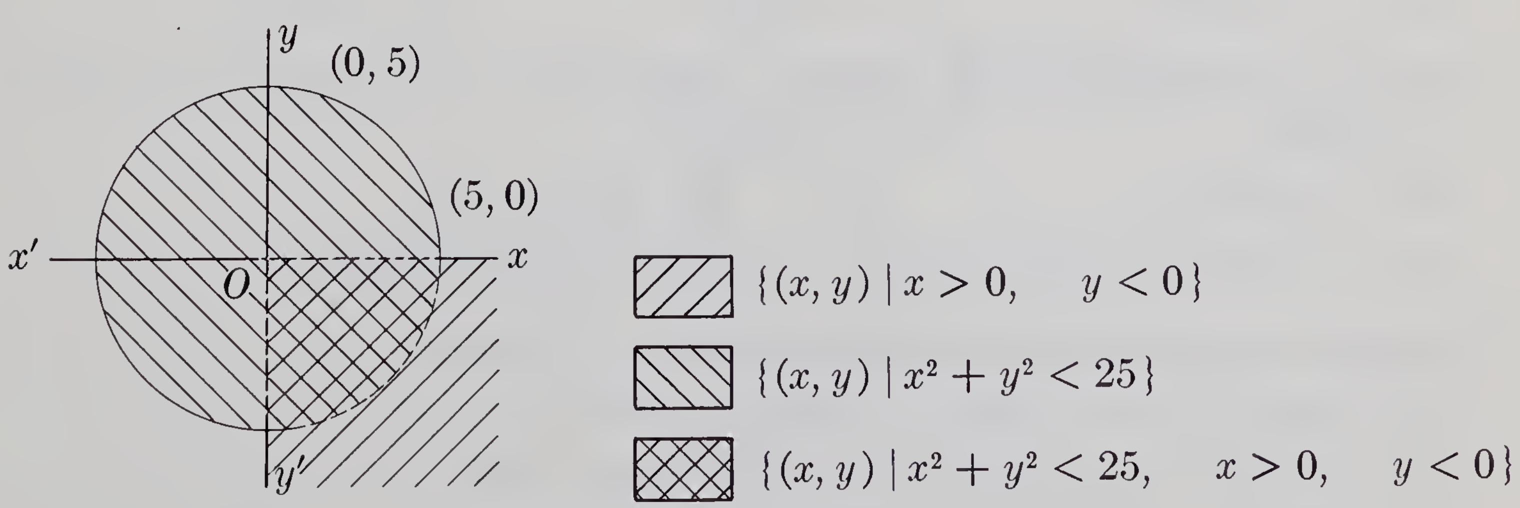


Figure 3.6

EXERCISE 3.1

1. State equations for circles with centre at the origin and the following radii.

   (a) 4 (b) $\frac{3}{5}$ (c) $2\sqrt{2}$ (d) $\frac{1}{\sqrt{3}}$ (e) $\frac{1}{4}\sqrt{7}$

2. Find equations for the circles, centre $(0, 0)$, which pass through the following points.

   (a) $(5, 12)$ (b) $(1, 2)$ (c) $(0, 3\sqrt{3})$
   (d) $(-2, -5)$ (e) $(4, 5)$ (f) $(-6, -8)$

3. What are the radii of the circles represented by the following equations?

   (a) $x^2 + y^2 = 49$ (b) $x^2 + y^2 = 12$
   (c) $4x^2 + 4y^2 = 81$ (d) $25x^2 + 25y^2 = 18$

4. For each of the following, find the intercepts, domain, and range, and sketch the graph $(x, y \in Re)$.

   (a) $x^2 + y^2 = 4$ (b) $3x^2 + 3y^2 = 1$
   (c) $25x^2 + 25y^2 = 64$ (d) $x^2 + y^2 = 6$

5. Apply the three tests of Section 3.1 for symmetry to determine what symmetry, if any, is possessed by the graphs of each of the following.

   (a) $4x^2 + 9y^2 = 36$ (b) $y = 3x^2$
   (c) $y = x^3$ (d) $3x + 5y = 15$
   (e) $y^2 = 4x - 16$ (f) $x^2 - y^2 = 4$
   (g) $y = |x|$ (h) $xy = 8$

6. For each of the following points, determine whether it is outside, inside, or on the circle defined by $x^2 + y^2 = 25$.

   (a) $(-5, 0)$ (b) $(-1, -4.5)$ (c) $(4, -3)$
   (d) $(-1, 2\sqrt{6})$ (e) $(1, 2\sqrt{6})$ (f) $(4, 3.5)$

7. Sketch the graphs of the relations defined by the following inequalities $(x, y \in Re)$.

   (a) $x^2 + y^2 > 1$ (b) $x^2 + y^2 \leq 9$
   (c) $x^2 + y^2 \leq 4,\ x < 0,\ y > 0$ (d) $4x^2 + 4y^2 < 9,\ y < x$

8. Sketch the graphs of the following relations $(x, y \in Re)$.

   (a) $\{(x, y) \mid x^2 + y^2 \geq 16\} \cap \{(x, y) \mid y > 2x - 4\}$
   (b) $\{(x, y) \mid x^2 + y^2 < 25\} \cap \{(x, y) \mid 4x^2 + 4y^2 > 9\}$
   (c) $\{(x, y) \mid x^2 + y^2 < 1\} \cup \{(x, y) \mid x^2 + y^2 \geq 4\}$
   (d) $\{(x, y) \mid x^2 + y^2 \leq 25\} \cap \{(x, y) \mid x^2 + y^2 \geq 4\} \cap \{(x, y) \mid x < 0,\ y < 0\}$

9. For each of the following determine the intercepts, the domain, and the range; discuss the symmetry and sketch the graph $(x, y \in Re)$.
   (a) $\{(x, y) \mid x^2 + y^2 = 16, \ |x| > 1\}$
   (b) $\{(x, y) \mid x^2 + y^2 - 4y = 5\}$ (Complete the square of the terms in y.)
   (c) $\{(x, y) \mid x^2 + 2x + y^2 = 8\}$

3.2. Circle Problems Solved by Analytic Methods (Supplementary)

Example 1. A chord of the circle $x^2 + y^2 = 65$ is represented by $3x - y + 5 = 0$. Find the length of the chord.

Solution: If

$$3x - y + 5 = 0\,,$$

then

$$y = 3x + 5\,.$$

Therefore, at the end-points of the chord,

$$x^2 + (3x + 5)^2 = 65\,,$$

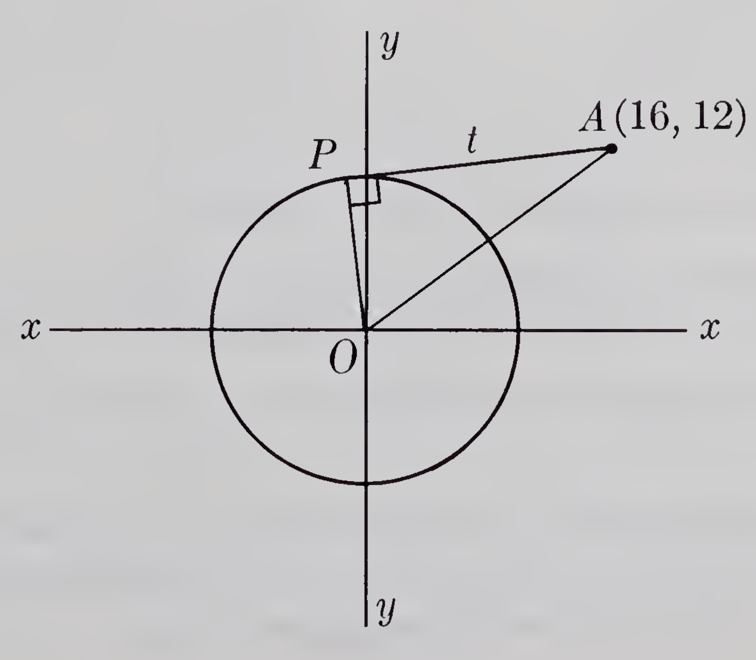
or

$$\begin{aligned} 10x^2 + 30x - 40 &= 0\,, \\ x^2 + 3x - 4 &= 0\,, \\ (x + 4)(x - 1) &= 0\,. \end{aligned}$$

Hence, $x = -4$ or $x = 1$, and $y = -7$ or $y = 8$. Therefore, the endpoints of the chord are $(-4, -7)$ and $(1, 8)$. The length of the chord is

$$\sqrt{(1 + 4)^2 + (8 + 7)^2} = 5\sqrt{10}\,.$$

Example 2. Find the length of the tangent drawn from the point $A\,(16, 12)$ to the circle whose equation is $x^2 + y^2 = 100$.



Solution: Suppose that t units is the length of the tangent AP where P is the point of contact. $\angle APO = 90°$;
thus,

$$\begin{aligned} t^2 &= AO^2 - OP^2 \\ &= (16 - 0)^2 + (12 - 0)^2 - 100 \\ &= 256 + 144 - 100 \\ &= 300\,. \end{aligned}$$

Therefore,

$$t = 10\sqrt{3}\,.$$

The length of the tangent AP is $10\sqrt{3}$ units.

EXERCISE 3.2

1. A chord of the circle $x^2 + y^2 = 36$ joins the point of intersection of the circle and the x-axis to a point of intersection of the circle and the y-axis.

   Find the length of the perpendicular from the centre of the circle to such a chord.

2. A circle, centre the origin, passes through $A\,(2, \sqrt{21})$ and $B(-\sqrt{21}, 2)$. Find an equation of the circle and show that the perpendicular bisector of chord AB passes through the centre of the circle.

3. For each of the following circles, the co-ordinates of the midpoint of a chord are given. Find the length of the chord and an equation of the chord.

   (a) $x^2 + y^2 = 25$ $(0, 4)$ (b) $x^2 + y^2 = 25$ $(1, 2)$

   (c) $4x^2 + 4y^2 = 57$ $(-2, \frac{5}{2})$ (d) $x^2 + y^2 = 16$ $(5, 6)$

4. There is no solution in question 3(d). Explain.

5. In each of the following, the equation of a circle and the co-ordinates of a point not on the circle are given. Find the length of the tangent from the given point.

   (a) $x^2 + y^2 = 25$ $(6, 5)$ (b) $4x^2 + 4y^2 = 81$ $(6.5, 7)$

   (c) $x^2 + y^2 = 3$ $(-3, 7)$ (d) $x^2 + y^2 = 169$ $(11, 20)$

   (e) $x^2 + y^2 = 100$ $(3, 4)$

6. There is no solution in question 5(e). Explain.

7. Develop a formula for the length of a tangent from $A\,(x_1, y_1)$ to the circle defined by $x^2 + y^2 = r^2$. Is there any restriction on x_1 and y_1? Explain.

8. Use the result of question (7) to calculate the lengths of the tangents in question (5).

9. A and A' are the points of intersection of the circle represented by $x^2 + y^2 = r^2$ and the x-axis. If $P\,(m, n)$ is any point on the circle, prove that angle APA' is a right angle. State this result in general terms.

10. Find an equation for the circle with centre $(2, 1)$ and radius 5 units.

11. For the circle in question (10), find the length of the tangent drawn to the circle from the point $(-5, 7)$.

3.3. The Parabola

In an earlier course, we have studied the relation

$$P = \{(x, y) \mid y = ax^2, \quad x \in Re, \quad a > 0\}.$$

Let us sketch the graph of P by determining the intercepts, domain, range, and any symmetry that exists.

Intercepts

In the relation

$$y = ax^2,$$

if

$$y = 0,$$

then

$$x = 0,$$

and if

$$x = 0,$$

then

$$y = 0.$$

The only x-intercept is zero and the only y-intercept is zero. The origin lies on the graph of P and is the only point of either axis to do so.

Domain

Since $y = ax^2$, there is a real value of y for all $x \in Re$. The domain is Re.

Range

Since $x^2 = \frac{y}{a}$, then $x = \pm\sqrt{\frac{y}{a}}$. For real values of x, we have $y \geq 0$. The range is $\{y \mid y \in Re, y \geq 0\}$. It now appears that the graph is confined to the region shaded in Figure 3.7.

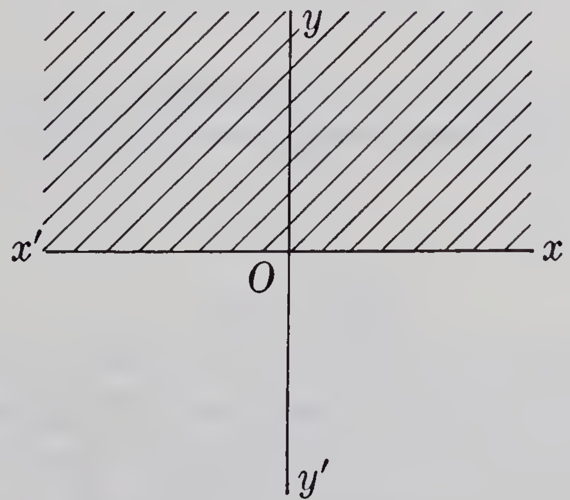


Figure 3.7

Symmetry

In $y = ax^2$, when x is replaced by $-x$, the equation is unchanged, and therefore the graph is symmetrical with respect to the y-axis. Replacing y by $-y$ or replacing both x by $-x$ and y by $-y$ changes the equation, and therefore, symmetry with respect to the x-axis and symmetry with respect to the origin do not exist.

For each point plotted in the first quadrant, we may plot its reflection in the y-axis. That is, each point (x, y) in the first quadrant has an image or reflection $(-x, y)$ in the second quadrant. The graph, which is shown in Figure 3.8, is a parabola opening upward, with its vertex at the origin and its axis of symmetry is the y-axis.

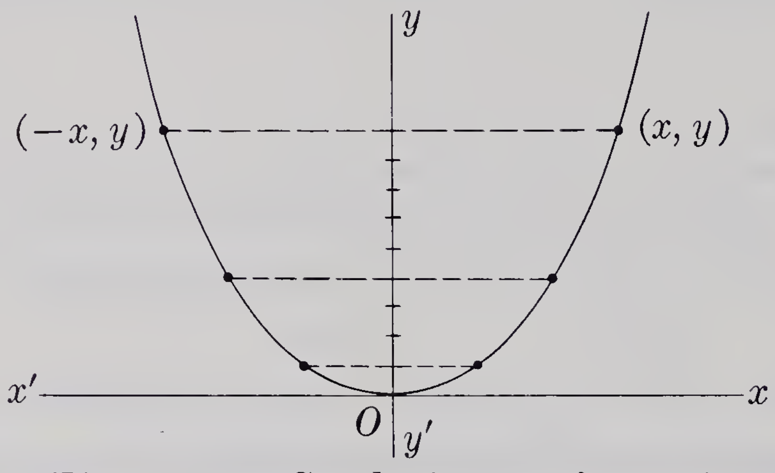


Figure 3.8. Graph of $y = ax^2$, $a > 0$

Now suppose $a < 0$. The discussion above may be repeated without change except for the fact that the range is $\{y \mid y \leq 0, y \in Re\}$ since $x = \pm\sqrt{\dfrac{y}{a}}$ and $a < 0$. The graph is a *parabola* opening downward, vertex at the origin and the y-axis as the axis of symmetry. See Figure 3.9.

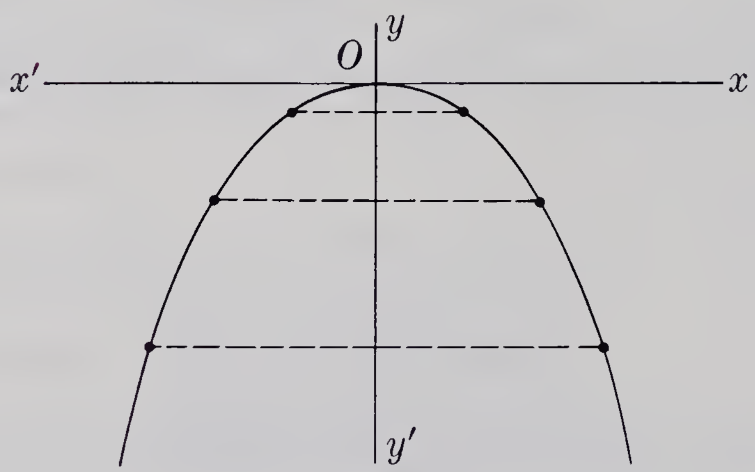


Figure 3.9

Imagine a line passing through a fixed point in space and tangent to a circle whose plane is perpendicular to the line joining its centre to the fixed point (Figure 3.10). The locus of points on all such lines through the fixed point is a surface called a right circular cone.

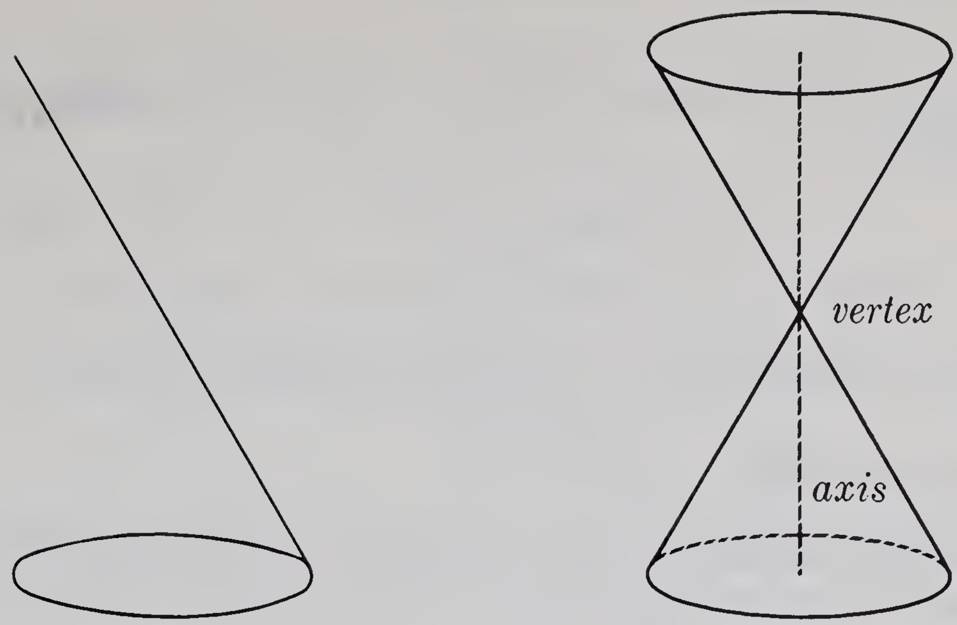


Figure 3.10

We say that the cone is a surface generated by a line passing through a fixed point (vertex) and touching a circle whose plane is perpendicular to the line (axis) joining its centre to the fixed point. As shown in Figure 3.10, the cone consists of two nappes, one above and one below the vertex.

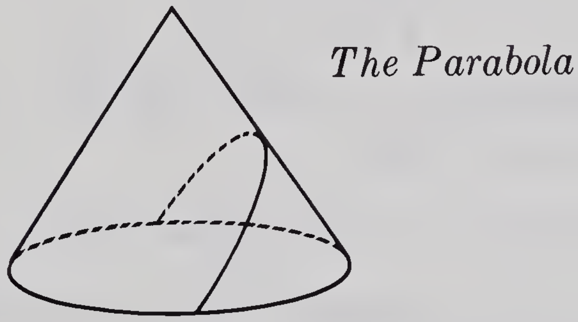


Figure 3.11

The curve of intersection of a right circular cone by a plane is called a conic section or, sometimes, a conic. If the cutting plane is parallel to the generating line (Figure 3.11), the conic is called a parabola. This is the curve we have studied from an analytic point of view in this section. Under the assumption of negligible air resistance, any ball thrown or kicked is found to follow a parabolic path. The path of a projectile or missile near the surface of the earth is also parabolic. If a load is uniformly distributed along the bridge, the cable of a suspension bridge assumes a parabolic form.

The laminated arches that support the roofs in many modern buildings and also the supporting arches of bridges are often parabolic.

EXERCISE 3.3

For the relations defined by each of the following, find the intercepts, domain, range, and symmetry, and sketch the graph.

1. $y = -x^2$	2. $y = 4x^2$	3. $y = \frac{1}{4}x^2$
4. $2y = 5x^2$	5. $x^2 = -8y$	6. $x^2 = 16y$

Use the solutions for questions (1) to (6) above to sketch the regions defined by the following inequalities.

7. $y > 4x^2$
8. $y > 4x^2, \quad x > 0$
9. $x^2 < -8y$
10. $x^2 < -8y, \quad x < 0$
11. $y > \frac{1}{4}x^2, \quad y < 4x^2$
12. $y < 4x^2, \quad -8y < x^2$
13. The point with co-ordinates $(8, -8)$ is on the graph defined by
$$-ky = x^2 .$$
What is the value of k?
14. Describe the graph defined by $x^2 = my$ when m is replaced by zero. (This graph is called a degenerate parabola.) Can you give a geometric interpretation by referring to Figure 3.9?
15. By analogy to the graph of
$$y = 4x^2 ;$$
that is,
$$y - 0 = 4(x - 0)^2 ,$$
sketch the graph of
$$y - 2 = 4(x - 1)^2 .$$
16. Sketch the graph defined by $y = 3x^2 - 12x + 7$. First reduce the equation to the form $y - d = a(x - m)^2$.
17. Repeat question (16) for $y = -4x^2 - 24x - 34$.
18. As a increases from zero through positive values, describe the change that takes place in the graph defined by $y = ax^2$.

3.4. The Ellipse

Let us sketch the graph of the relation E where
$$E = \{(x, y) \mid 4x^2 + 9y^2 = 36, \quad x, y \in Re\} .$$

Intercepts

In
$$4x^2 + 9y^2 = 36 ,$$
if
$$y = 0 ,$$
then
$$x^2 = 9 ,$$
and
$$x = +3 \quad \text{or} \quad -3 .$$
The x-intercepts are ± 3 and the points $(3, 0)$ and $(-3, 0)$ are on the graph.

In the equation

$$4x^2 + 9y^2 = 36\,,$$

if

$$x = 0\,,$$

then

$$y^2 = 4\,,$$

and

$$y = +2 \quad \text{or} \quad -2\,.$$

The y-intercepts are ± 2 and the points $(0, 2)$ and $(0, -2)$ are on the graph.

Domain

Since

$$4x^2 + 9y^2 = 36\,,$$

then

$$9y^2 = 36 - 4x^2\,,$$

and

$$3y = \pm\sqrt{36 - 4x^2}\,.$$

For real values of y,

$$36 - 4x^2 \geq 0\,,$$

so that

$$4x^2 \leq 36\,,$$

and

$$x^2 \leq 9\,,$$

and

$$|\,x\,| \leq 3\,.$$

The domain is $\{x \mid x \in Re,\ -3 \leq x \leq 3\}$.

Range

Since

$$4x^2 + 9y^2 = 36\,,$$

then

$$4x^2 = 36 - 9y^2\,,$$

and

$$2x = \pm\sqrt{36 - 9y^2}\,.$$

For real values of x,

$$36 - 9y^2 \geq 0\,,$$

so that

$$9y^2 \leq 36\,,$$

and

$$y^2 \leq 4\,,$$

and

$$|\,y\,| \leq 2\,.$$

The range is $\{y \mid y \in Re,\quad -2 \leq y \leq 2\}$.

We now know that the graph is restricted to the shaded rectangular region in Figure 3.12.

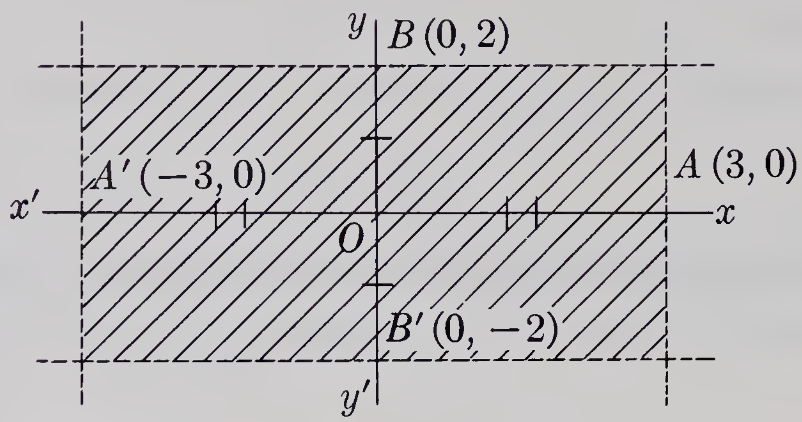


Figure 3.12

Symmetry

The equation $4x^2 + 9y^2 = 36$ is unchanged when x is replaced by $-x$. The graph is symmetric with respect to the y-axis.

The equation is unchanged when y is replaced by $-y$. The graph is symmetric with respect to the x-axis.

The equation is also unchanged when x is replaced by $-x$ *and* y by $-y$. The graph must be symmetric with respect to the origin.

It will be sufficient to plot points in the first quadrant. Points in the other three quadrants may be determined by symmetry (Figure 3.13). A table of values precedes the figure. Some values of y in the table are approximate. The graph is called an *ellipse* with centre at the origin.

x	y
0	2
1	1.9
2	1.5
2.5	1.1
3	0

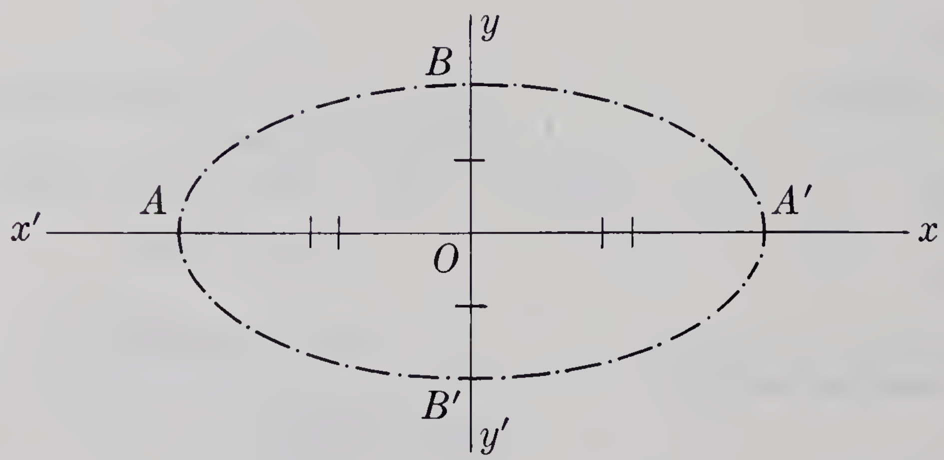


Figure 3.13

In the ellipse, the line segment AA' is called the *major axis* and the line segment BB' the *minor axis*. The line segment OA or OA' is called the *semi-major axis* and the line segment OB or OB' *semi-minor axis*. The point O is called the *centre* and the points A and A', which are the end-points of the major axis, are called the *vertices* of the ellipse.

Consider the equation $4x^2 + 9y^2 = 36$. If both members are divided by 36, we obtain

$$\frac{x^2}{9} + \frac{y^2}{4} = 1\,,$$

or

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1\,.$$

Note that the denominators on the left side are the squares of the x-and y-intercepts. Similarly, any relation whose defining equation is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

(where a and b are unequal positive real numbers and $a > b$) has as its graph the ellipse shown in Figure 3.14.

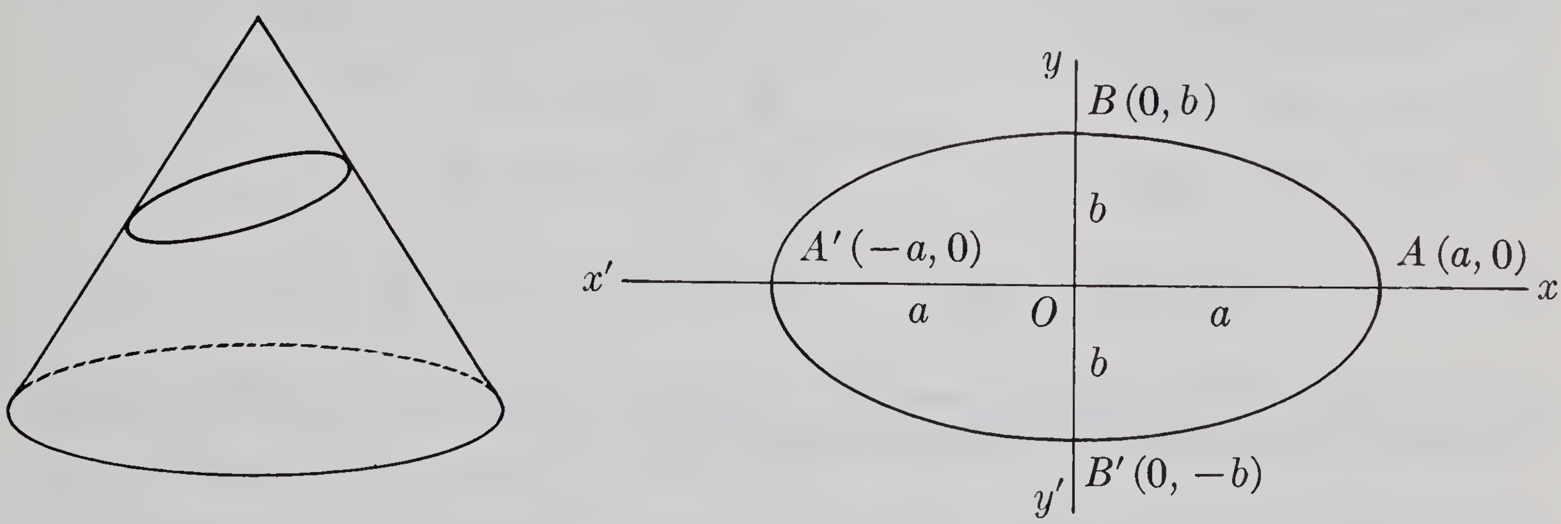


Figure 3.14

We say that the semi-major axis is a units in length and that the semi-minor axis is b units in length. The origin is the centre of the ellipse. An ellipse is sometimes referred to as a central conic.

The Ellipse

When a cutting plane cuts entirely across one nappe of a cone, the conic section obtained is the ellipse.

The orbits of the planets, including our own, are ellipses. Man-made satellites travel around the earth in elliptical orbits. The question of elliptical orbits for the planets (versus circular orbits) was the subject of bitter debate during the Renaissance. Consult an encyclopedia or other reference work for articles on Galileo and Kepler.

The arches of bridges are frequently constructed in the form of semi-ellipses.

Elliptical gears are used in machines such as power machines, where slow and powerful motion is needed during a part of each revolution.

EXERCISE 3.4

For the relations defined by each of the following, find the intercepts, domain, range, symmetry, and sketch the graph.

1. $16x^2 + 25y^2 = 400$
2. $12x^2 + 25y^2 = 300$
3. $x^2 + 9y^2 = 9$
4. $2x^2 + 5y^2 = 40$
5. $\dfrac{x^2}{3} + y^2 = 1$
6. $\dfrac{x^2}{64} + \dfrac{y^2}{9} = 1$
7. For each of the ellipses sketched in questions (1) to (6), state
   (a) the lengths of the semi-major and semi-minor axes,
   (b) the co-ordinates of the vertices.

Use the solution for questions (1) to (6) above to sketch the regions defined by the following inequalities.

8. $16x^2 + 25y^2 < 400$
9. $x^2 + 9y^2 > 9$
10. $16x^2 + 25y^2 < 400, x < 0$
11. $2x^2 + 5y^2 \geq 40, \quad x > 0, \quad y < 0$
12. $x^2 > y, \qquad 16x^2 + 25y^2 < 400$
13. $x^2 + y^2 > 1, \quad \dfrac{x^2}{64} + \dfrac{y^2}{9} < 1$
14. The point whose co-ordinates are $(2, -3)$ is on the graph of the relation defined by $9x^2 + ky^2 = 54$. What is the value of k?
15. Describe the ellipse with centre at the origin and both semi-axes equal to 4. What is the equation of this ellipse?
16. Are the points whose co-ordinates are given outside, inside, or on the ellipse defined by
$$4x^2 + 9y^2 = 36?$$
   (a) $(-2, -1)$ (b) $(3, -2)$ (c) $(-1, 1.5)$ (d) $\left(-2, -\dfrac{2\sqrt{5}}{3}\right)$
17. By analogy to the graph of
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$
   that is,
$$\frac{(x-0)^2}{a^2} + \frac{(y-0)^2}{b^2} = 1$$
   sketch the graph of $\dfrac{(x-3)^2}{25} + \dfrac{(y-1)^2}{16} = 1\,.$
18. Repeat question (17) for $\dfrac{(x+5)^2}{4} + \dfrac{(y+1)^2}{1} = 1\,.$

3.5. The Hyperbola

Let us sketch the graph of relation H where

$$H = \{(x, y) \mid 4x^2 - 9y^2 = 36, \quad x, y \in Re\}.$$

Intercepts

In the equation

$$4x^2 - 9y^2 = 36,$$

if

$$y = 0,$$

then

$$x^2 = 9,$$

and

$$x = +3 \text{ or } -3.$$

The x-intercepts are ± 3 and the points $(3, 0)$ and $(-3, 0)$ are on the graph.

In the equation

$$4x^2 - 9y^2 = 36,$$

if

$$x = 0,$$

then

$$y^2 = -4.$$

There is no real value of y that satisfies this equation. There are no y-intercepts and the graph has no points on the y-axis.

Domain

Since

$$4x^2 - 9y^2 = 36,$$

then

$$9y^2 = 4x^2 - 36,$$

and

$$3y = \pm\sqrt{4x^2 - 36}.$$

For real values of y,

$$4x^2 - 36 \geq 0,$$

so that

$$4x^2 \geq 36,$$

and

$$x^2 \geq 9,$$

and

$$|x| \geq 3.$$

The domain is $\{x \mid x \in Re, x \geq 3 \text{ or } x \leq -3\}$.

Range

Since

$$4x^2 - 9y^2 = 36\,,$$

then

$$4x^2 = 9y^2 + 36, \qquad \text{and} \qquad 2x = \pm\sqrt{9y^2 + 36}\,.$$

For real values of x,

$$9y^2 + 36 \geq 0\,.$$

Since $9y^2 + 36 \geq 0$ for all $y \in Re$, the range is Re. We now know that the graph is restricted to the shaded region in Figure 3.15.

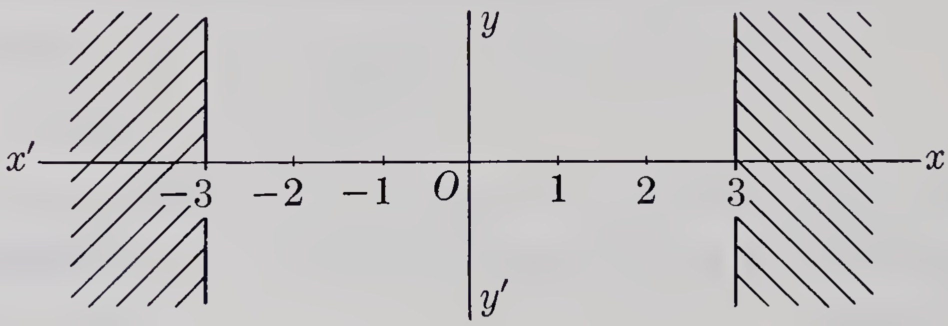


Figure 3.15

Symmetry.

Since the equation

$$4x^2 - 9y^2 = 36$$

is unchanged when x is replaced by $-x$, the graph is symmetric with respect to the y-axis. Since the equation is unchanged when y is replaced by $-y$, the graph is symmetric with respect to the x-axis. The equation is also unchanged when x is replaced by $-x$ *and* y is replaced by $-y$ so that the graph must be symmetric with respect to the origin.

It is sufficient to plot points in the first quadrant. Points in the other three quadrants may be determined by symmetry. A table of values precedes Figure 3.16. Some values of y in the table are approximate.

x	y
3	0
3.5	1.2
4	1.8
5	2.7

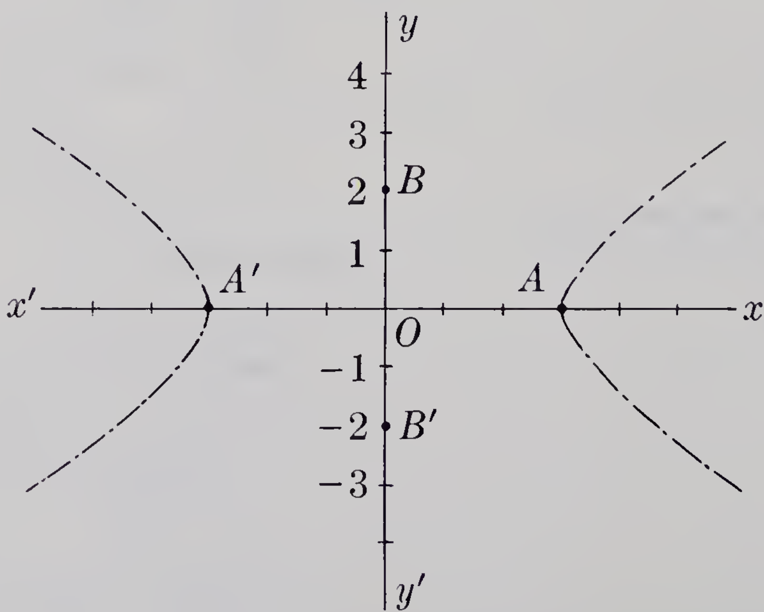


Figure 3.16

The graph is called a *hyperbola* with centre at the origin. If we divide both members of the defining equation by 36, we obtain

$$\frac{x^2}{9} - \frac{y^2}{4} = 1, \quad \text{or} \quad \frac{x^2}{3^2} - \frac{y^2}{2^2} = 1.$$

Note that, as in the case of the ellipse, the denominator of the first term is the square of the x-intercept.

The line segment AA' is called the *transverse axis* of the hyperbola and the points A and A' are the vertices of the curve.

The denominator of the second term locates for us the points $B(0, 2)$ and $B'(0, -2)$ on the y-axis. While there are no y-intercepts, the segment BB' of the y-axis is called the *conjugate axis* of the hyperbola.

Similarly, any relation whose defining equation is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

(where a and b are positive real numbers) has as its graph the hyperbola shown in Figure 3.17.

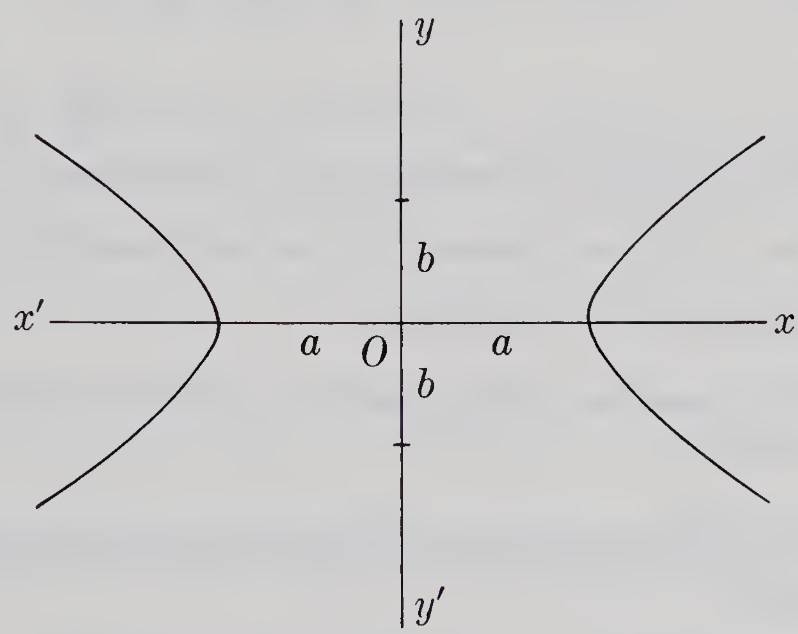


Figure 3.17

semi-transverse axis is a units in length and the semi-conjugate axis units in length. The hyperbola, like the ellipse and the circle, is a central conic.

The Hyperbola

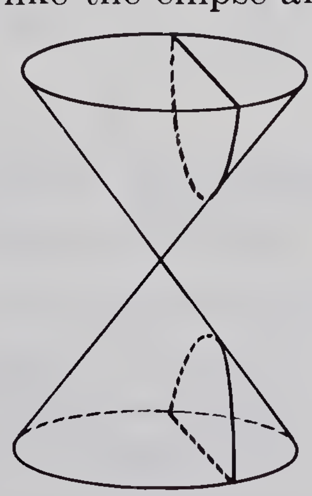


Figure 3.18

If both nappes of a right circular cone are cut by the same plane, the resulting conic section is a hyperbola. The parabola, ellipse, and hyperbola were named and studied thoroughly by the ancient Greek geometers. Using synthetic methods they discovered many properties of these curves, including the definitions of the conics which we shall use in the next chapter.

The hyperbola is the graph of many relations in science, for example Boyle's Law, which relates the volume of a gas to its pressure if the temperature is constant.

The properties of the hyperbola are important in modern navigation systems such as Loran. Further reference will be made to this in the next chapter.

EXERCISE 3.5

For the relations defined by each of the following, find the intercepts, domain, range, symmetry, and sketch the graph.

1. $16x^2 - 25y^2 = 400$
2. $25x^2 - 4y^2 = 100$
3. $5x^2 - y^2 = 20$
4. $\dfrac{x^2}{20} - \dfrac{y^2}{49} = 1$
5. $4x^2 - y^2 = 1$
6. $x^2 - y^2 = 25$
7. For each of the hyperbolas in questions (1) to (6), state
   (a) the lengths of the semi-transverse and semi-conjugate axes,
   (b) the co-ordinates of the vertices.
8. The hyperbola in question (6) is called an equilateral hyperbola. Suggest the reason for this name.

Use the solutions for questions (1) to (6) above to sketch the regions defined by the following inequalities.

9. $16x^2 - 25y^2 < 400$
10. $5x^2 - y^2 > 20$
11. $16x^2 - 25y^2 < 400, \quad x, y < 0$
12. $\dfrac{x^2}{20} - \dfrac{y^2}{49} > 1, \quad x > 0$
13. $4x^2 - y^2 < 1, \quad x^2 + y^2 < 1$
14. $x^2 < y, \quad x^2 - y^2 < 25$
15. The point whose co-ordinates are $(\frac{3}{2}, -2\sqrt{2})$ is on the graph of the relation defined by $4x^2 - ky^2 = 1$. What is the value of k?
16. Describe the graph defined by

$$\frac{x^2}{16 - m} + \frac{y^2}{9 - m} = 1$$

when (a) $m < 9$, (b) $9 < m < 16$.

17. By comparison with the graph of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{that is,} \quad \frac{(x-0)^2}{a^2} - \frac{(y-0)^2}{b^2} = 1\,,$$

sketch the hyperbola defined by

$$\frac{(x-3)^2}{25} - \frac{y^2}{16} = 1\,.$$

18. Repeat question (17) for

$$\frac{(x+4)^2}{49} - \frac{(y-5)^2}{64} = 1\,.$$

3.6. Another Defining Equation for the Hyperbola

Example 1. Sketch the graph of the relation

$$H_1 = \{(x, y) \mid xy = 36, x, y \in Re\}\,.$$

Solution:

Intercepts. Replacing either x or y by 0 in $xy = 36$ does not produce a value for the other variable. There are no x- or y-intercepts.

Domain. Since

$$xy = 36\,,$$

$$y = \frac{36}{x}\,.$$

Real values of y are produced by all $x \in Re$ except zero. The domain is $\{x \mid x \in Re, x \neq 0\}$.

Range. Since

$$xy = 36\,,$$

$$x = \frac{36}{y}\,.$$

Real values of x correspond to all $y \in Re$ except zero. The range is $\{y \mid y \in Re, y \neq 0\}$.

Symmetry. The equation $xy = 36$ is changed when x is replaced by $-x$. The graph is not symmetrical with respect to the y-axis. The equation is also changed when y is replaced by $-y$. The graph is not symmetric with respect to the x-axis. The equation is unchanged when x is replaced by $-x$ *and* y is replaced by $-y$. The graph is symmetric with respect to the origin.

Since $xy = 36$, x and y are both positive or both negative. Symmetry about the origin means that each point plotted in the first quadrant has its reflection in the third quadrant. The required graph, preceded by a table of values, is shown in Figure 3.19.

x	y
3	12
4	9
6	6
8	4.5
9	4

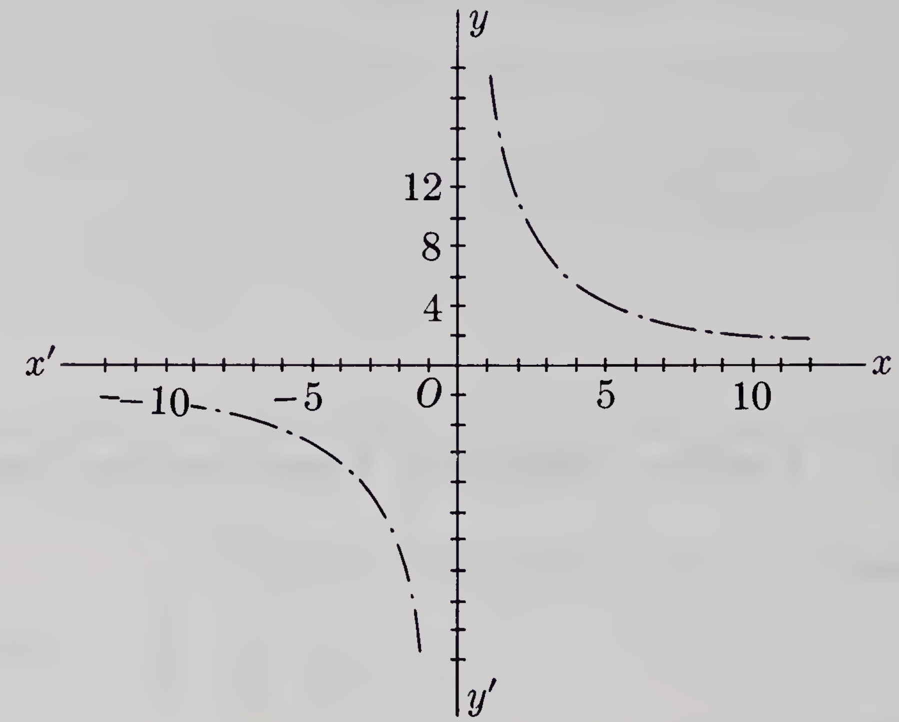


Figure 3.19

The graph is a hyperbola. It is actually an equilateral hyperbola such as that defined by $x^2 - y^2 = 25$. The transverse axis lies on the line lying in the first and third quadrants and bisecting the angle between the co-ordinate axes.

Example 2. Sketch the graph of the relation

$$L = \{(x, y) \mid 4x^2 - 9y^2 = 0, \quad x, y \in Re\}.$$

Solution:

Intercepts. In the equation $4x^2 - 9y^2 = 0$,

if

$$y = 0,$$

then

$$x = 0.$$

and

if

$$x = 0,$$

then

$$y = 0.$$

The x- and y-intercepts are both zero. The origin is on the graph.

Domain. If

$$4x^2 - 9y^2 = 0,$$

then

$$9y^2 = 4x^2,$$

and

$$3y = \pm 2x.$$

For real values of y, we see that $2x$ must be real and this is true for all $x \in Re$. The domain is Re.

Range. Similarly, $2x = \pm 3y$. The range is Re.

Symmetry. Applying the three tests for symmetry to the equation $4x^2 - 9y^2 = 0$ shows that the graph is symmetric with respect to both axes and the origin. This symmetry allows us to plot points in the second, third, and fourth quadrants as reflections of first quadrant points.

x	y
0	0
1	$\frac{2}{3}$
2	$\frac{4}{3}$
3	2

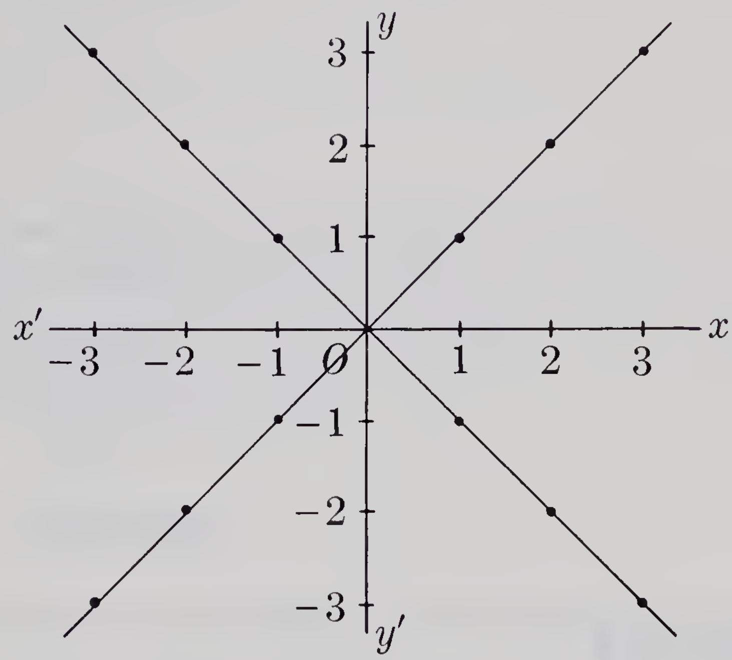


Figure 3.20

The graph, shown in Figure 3.20, is a pair of straight lines through the origin.
Note: The defining equation of the relation L may be written as

$$(2x - 3y)(2x + 3y) = 0 .$$

Therefore,

$$2x - 3y = 0 \quad \text{or} \quad 2x + 3y = 0 .$$

Hence, if the co-ordinates of a point (x, y) satisfy the equation $4x^2 - 9y^2 = 0$ they must satisfy one or the other of $2x - 3y = 0$ or $2x + 3y = 0$. Conversely, it must also be true that a point whose co-ordinates satisfy one or the other of $2x - 3y = 0$ or $2x + 3y = 0$ has co-ordinates that satisfy $4x^2 - 9y^2 = 0$. Therefore, the graph of L is the graph of

$$\{(x, y) \mid 2x - 3y = 0 \quad \text{or} \quad 2x + 3y = 0\} .$$

The graph of L is then the pair of straight lines defined by

$$2x - 3y = 0 \quad \text{and} \quad 2x + 3y = 0 .$$

Such a pair of straight lines is called a degenerate conic. The cutting plane passes through the vertex of the cone.

Example 3. Sketch the graph of the relation

$$\{(x, y) \mid x^2 - 5x + 6 = 0\}.$$

Solution: The defining equation may be rewritten as

$$(x - 2)(x - 3) = 0.$$

Therefore, $x - 2 = 0$ or $x - 3 = 0$. The graph (Figure 3.21) is the two parallel straight lines defined by

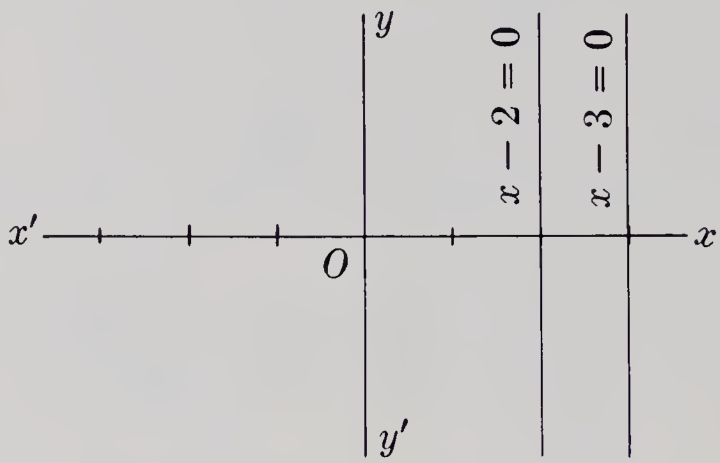


Figure 3.21

EXERCISE 3.6

Sketch the graphs of the relations defined by the following equations.

1. $xy = 4$
2. $xy = -4$
3. $xy = \frac{25}{2}$
4. $4x^2 - y^2 = 0$
5. $x^2 - 8x + 12 = 0$
6. $x^2 - xy - 12y^2 = 0$
7. $8x^2 + 2xy - 15y^2 = 0$
8. $(x - 3)^2 + 3(x - 2) + 2 = 0$
9. $3x^2 + 8xy - 35y^2 = 0$
10. $y^2 - 4y - 45 = 0$
11. Compare the graph in question (3) with that of $x^2 - y^2 = 25$ (question (6) in Exercise 3.5). What appears to be an analogous conclusion for the graphs of $$x^2 - y^2 = a^2 \quad \text{and} \quad xy = \frac{a^2}{2}?$$

Use the solutions for questions (1) to (10) to assist in the constructions of the graphs of the following inequalities.

12. $xy < 4$
13. $4x^2 - y^2 \geq 0$
14. $x - 6 < 0$ and $x - 2 > 0$
15. $x^2 - 8x + 12 > 0$
16. $y - 9 > 0$ or $y + 5 < 0$
17. $xy < 4$ and $x^2 + y^2 < 25$

18. Using the same set of axes, sketch the graphs of
$$4x^2 - y^2 = 0$$
and
$$4x^2 - y^2 = 4\,.$$

19. Repeat question (17) for $16x^2 - 25y^2 = 0$ and $16x^2 - 25y^2 = 400$. Can you suggest an alternative method for drawing the hyperbola? The lines defined by
$$4x \pm 5y = 0$$
are called the *asymptotes* of the hyperbola.

Chapter Summary

The equation of a circle · Analytic methods for circle problems (supplementary) · Graphs of the circle, parabola, ellipse, and hyperbola · Conics as sections of a right circular cone · The corresponding inequalities and their graphs · The equilateral hyperbola defined by $xy = k$ · Pairs of straight lines

REVIEW EXERCISE 3

1. State equations for circles with centre at the origin and the following radii.
   (a) 10 (b) $\frac{4}{9}$ (c) $\sqrt{5}$ (d) $\frac{1}{3}\sqrt{13}$

2. Find equations for the circles, centre (0, 0), that pass through the following points.
   (a) $(8, -6)$ (b) $(-2\sqrt{5}, 3)$ (c) $(\sqrt{14}, -2)$

3. Find the radii of the circles represented by the following equations.
   (a) $4x^2 + 4y^2 = 25$ (b) $x^2 + y^2 = 28$
   (c) $6x^2 + 6y^2 = 35$ (d) $50x^2 + 50y^2 = 1$

4. Apply the three tests for symmetry used in this chapter to determine what symmetry, if any, is possessed by the graphs of each of the following.
   (a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 25$
   (c) $y = 3x^2$ (d) $y = x^5$
   (e) $xy = -32$ (f) $y = |x| + 2$
   (g) $x^2 = 3y - 15$ (h) $4x^2 + 25y^2 = 1$

Sketch the graphs of the relations defined by each of the following.

5. $x^2 + y^2 = 100$
6. $50x^2 + 50y^2 = 81$
7. $x^2 + y^2 - 2y = 8$
8. $x^2 + y^2 < 9, x < 0, y > 0$
9. $y = -4x^2$
10. $x^2 = 10y$

11. $y > -4x^2, \quad x^2 + y^2 < 25$

12. $x^2 > -\frac{y}{4}, \quad y < 4x^2$

13. $x^2 + 4y^2 = 20$

14. $\frac{x^2}{81} + \frac{y^2}{9} = 1$

15. $x^2 + 4y^2 < 20, \quad x < 0, y < 0$

16. $\frac{x^2}{81} + \frac{y^2}{9} > 1, \quad x^2 + y^2 < 144$

17. $\frac{x^2}{16} - \frac{y^2}{100} = 1$

18. $x^2 - y^2 = 4$

19. $xy = -25$

20. $x^2 - y^2 < 4, \quad x^2 + y^2 > 1$

21. $x^2 - 6x + 5 = 0$

22. $x^2 - 6x + 9 = 0$

23. $64x^2 - 25y^2 = 0$

24. $3x^2 - 10xy - 8y^2 = 0$

25. $64x^2 - 25y^2 > 0, \quad x^2 + y^2 < 16$

26. $xy > 8, \quad \frac{x^2}{25} + \frac{y^2}{16} < 1$

If the point whose co-ordinates are indicated is on the graph of the relation whose defining equation is given, find the value of k.

27. $x^2 + y^2 = k, \quad (1, \sqrt{2})$

28. $x^2 = -ky, \quad (4, -1)$

29. $25x^2 + ky^2 = 300, \quad (2, 5\sqrt{2})$

30. $\frac{x^2}{40} - \frac{y^2}{50} = k, \quad (4\sqrt{10}, 2\sqrt{5})$

31. We have called a pair of straight lines, such as those defined by $4x^2 - 9y^2 = 0$, a degenerate conic. Under what geometric conditions would the degenerate conic be (a) a single straight line, (b) a single point?

32. Given the equation

$$\frac{x^2}{30 - m} + \frac{y^2}{m - 5} = 1,$$

find the set of values of m for which the equation defines
(a) an ellipse, (b) a hyperbola.

Chapter 4

GEOMETRIC DEFINITIONS OF CONICS · TANGENTS TO CONICS

4.1. Focus-directrix Definition of a Parabola

In Chapter 3, we sketched the conic sections by studying their defining equations. In this chapter, we shall develop the equations themselves, using certain fundamental properties of the conics as our definitions.

Consider the parabola defined by

$$x^2 = 4y, \qquad x \in Re\,.$$

$F(0, 1)$ is a point on the y-axis, one unit above the vertex, and

$$y + 1 = 0$$

represents a line perpendicular to the axis of symmetry and one unit below the vertex. The point P_1 with co-ordinates $(4, 4)$ is on the graph, as shown in Figure 4.1. The length of the line segment P_1F is

$$\sqrt{(4-0)^2 + (4-1)^2} = 5\,.$$

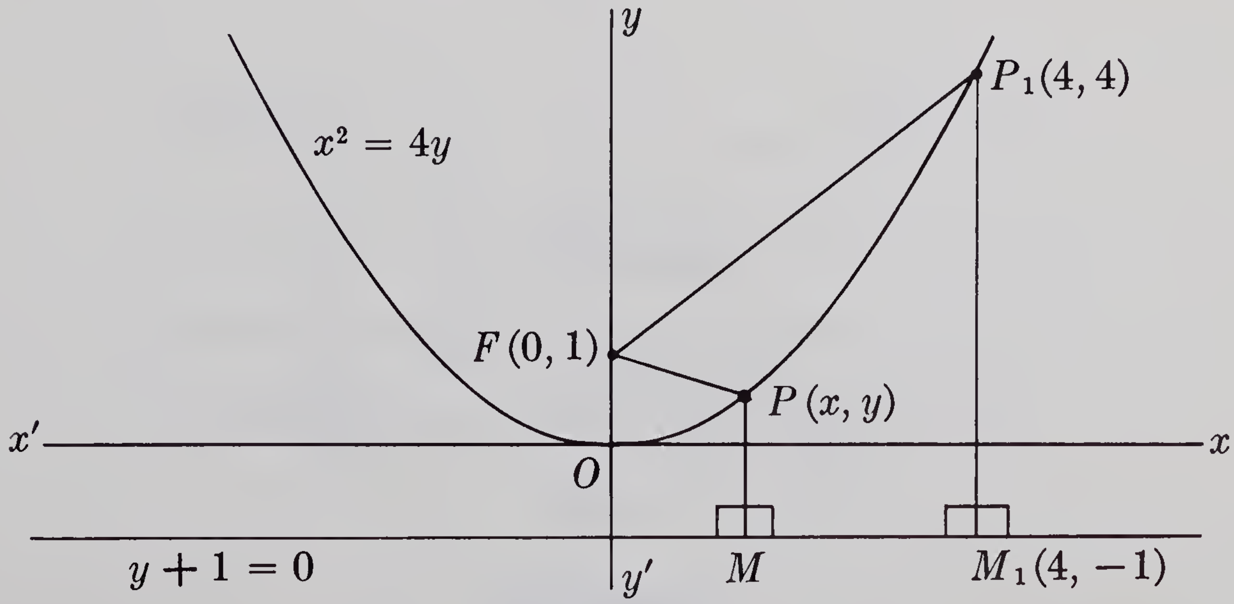


Figure 4.1

The length of the line segment P_1M_1, where M_1 is the foot of the perpendicular from P_1 to the line defined by

$$y + 1 = 0$$

is $4 + 1$ or 5. Hence,

$$P_1F = P_1M_1 .$$

Similarly, for any point $P(x, y)$ on the graph,

$$\begin{aligned} PF &= \sqrt{(x^2 - 0) + (y - 1)^2} \\ &= \sqrt{x^2 + y^2 - 2y + 1} \\ &= \sqrt{4y + y^2 - 2y + 1} && (\text{since } x^2 = 4y) \\ &= \sqrt{y^2 + 2y + 1} \\ &= |y + 1| \\ &= y + 1 && (\text{since } y \geq 0) \end{aligned}$$

and

$$PM = y + 1 .$$

Hence,

$$PF = PM$$

and

$$PF : PM = 1 : 1 .$$

The $\frac{1}{1}$ ratio implied here is called the eccentricity of the parabola. The point F is called the focus of the parabola and the line $y + 1 = 0$ is called the directrix of the parabola. We shall use the following formal definition for the parabola.

DEFINITION. A parabola is the set of points in a plane each of which is equidistant from a fixed point (the focus) and a fixed line (the directrix).

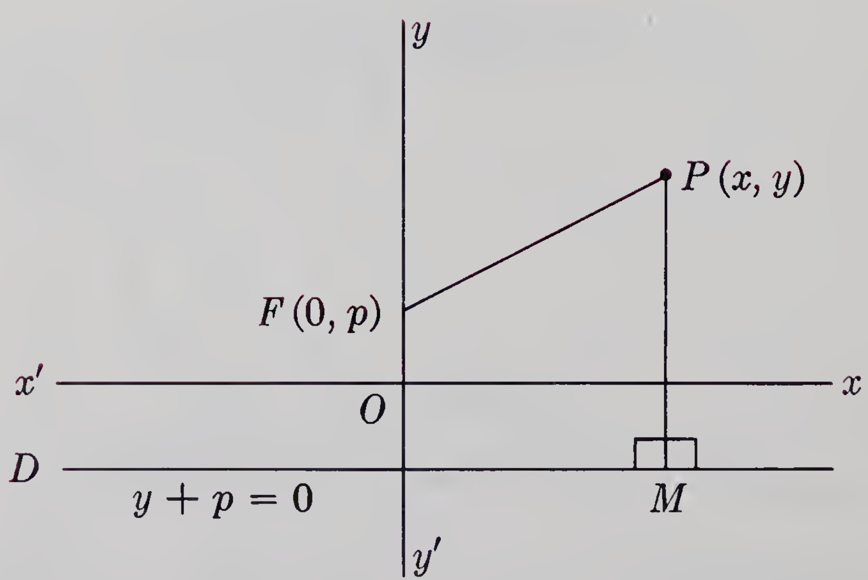


Figure 4.2

Theorem. An equation of the parabola with focus on the y-axis at $F(0, p)$, $p > 0$, and directrix the line defined by $y + p = 0$ is $x^2 = 4py$.

Proof: In Figure 4.2, if $P(x, y)$ is *any* point on the parabola, then

$$PF = PM\,, \qquad \text{(definition)}$$

thus,

$$\sqrt{(x-0)^2 + (y-p)^2} = y + p\,,$$

or

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2\,,$$

or

$$x^2 = 4py\,.$$

Conversely, if $P(x, y)$ is a point whose co-ordinates satisfy $x^2 = 4py$, then, on reversing the steps above, we obtain

$$x^2 + y^2 - 2py + p^2 = y^2 + 2py + p^2$$

and

$$\sqrt{x^2 + (y-p)^2} = \sqrt{(y+p)^2}\,.$$

Thus,

$$\begin{aligned} \sqrt{x^2 + (y-p)^2} &= |\,y + p\,| \\ &= y + p \qquad (y \geq 0 \text{ and } p > 0)\,. \end{aligned}$$

Therefore,

$$PF = PM\,.$$

Hence, $P(x, y)$ represents those and only those points that are equidistant from F and the line defined by $y + p = 0$. The equation of the parabola is therefore $x^3 = 4py$. This equation is called the standard form of the equation of the parabola opening upward with vertex at the origin and focus at $F(0, p)$, $p < 0$.

Similarly, it may be shown that the standard form of the equation of the parabola opening downward with vertex at the origin and focus at $F(0, p)$, $p < 0$, is $x^2 = 4py$.

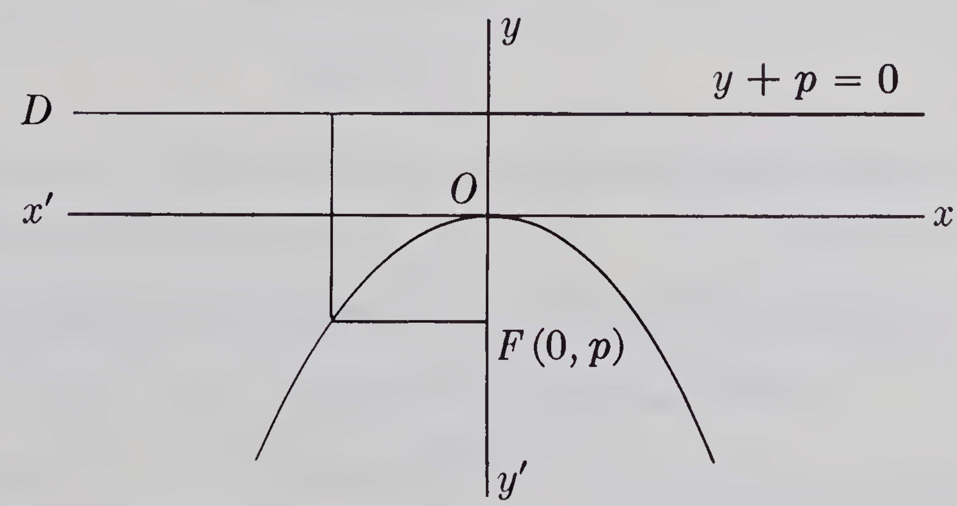


Figure 4.3

Note that the value of p may be obtained by calculating one quarter of the coefficient of y in the standard form of the equation, and with this value of p we may write the co-ordinates of the focus and the equation of the directrix.

Example 1. Find the co-ordinates of the focus and the equation of the directrix for the parabola defined by

$$x^2 = 25y\,.$$

Solution: From

$$x^2 = 25y\,,$$

$$p = \tfrac{25}{4}\,.$$

The co-ordinates of the vertex are $(0, \frac{25}{4})$ and the equation of the directrix is

$$y + \tfrac{25}{4} = 0\,,$$

or, with integral coefficients,

$$4y + 25 = 0\,.$$

Example 2. Find an equation of the parabola with vertex at the origin and focus at $F(0, -\frac{1}{8})$.

Solution: Since $p = -\frac{1}{8}$, replacing p by $-\frac{1}{8}$ in the equation $x^2 = 4py$, we obtain

$$x^2 = -\tfrac{1}{2}y\,,$$

or

$$2x^2 + y = 0\,.$$

EXERCISE 4.1

For the parabolas defined by the following equations state (i) the co-ordinates of the focus, (ii) the equation of the directrix, (iii) the ordinate of the point whose abscissa is -2.

1. $x^2 = 8y$
2. $5x^2 = 12y$
3. $x^2 = -\frac{15}{4}y$
4. $25x^2 = -y$

State equations for the parabolas defined by the following.

5. vertex $(0, 0)$, focus $(0, -4)$
6. vertex $(0, 0)$, directrix $y - 5 = 0$
7. focus $(0, 3)$, directrix $y + 3 = 0$
8. vertex $(0, 0)$, focus $(0, \frac{15}{2})$

Find values of p so that the parabola defined by

$$x^2 = 4py$$

will pass through the given point.

9. $(2, 1)$
10. $(-1, -1)$
11. $(6, -9)$
12. $(-6, -9)$

13. Prove the following theorem:

An equation of the parabola with focus on the x-axis at $F(p, 0)$, $p > 0$, and directrix the line defined by

$$x + p = 0$$

is

$$y^2 = 4px.$$

14. Prove the theorem:

An equation of the parabola with focus on the x-axis at $F(p, 0)$, $p < 0$, and directrix the line defined by

$$x + p = 0$$

is

$$y^2 = 4px.$$

For the parabolas defined by the following equations state (i) the co-ordinates of the focus, (ii) the equation of the directrix, and (iii) the abscissa of the point whose ordinate is 5.

15. $y^2 = 20x$

16. $4y^2 = -15x$

17. $y^2 = -\frac{1}{8}x$

18. $10y^2 = 25x$

State equations for the parabolas defined by the following.

19. vertex $(0, 0)$, directrix $x - 10 = 0$

20. vertex $(0, 0)$, focus $(-\frac{1}{3}, 0)$

21. focus $(\frac{7}{5}, 0)$, directrix $5x + 7 = 0$

22. vertex $(0, 0)$, focus $(-\frac{5}{8}, 0)$

Find values of p so that the parabola defined by

$$y^2 = 4px$$

will pass through the point given in each of questions 23-26.

23. $(-8, 4)$

24. $(-8, -4)$

25. $(10, 2\sqrt{5})$

26. $(-2, -\frac{1}{4})$

27. Find an equation for the parabola with vertex at the origin, focus on the x-axis, and opening to the right if the focus is on the line whose equation is

$$4x - 7y - 12 = 0.$$

28. Repeat question (27) given that the required parabola opens downward from the origin.

29. The line segment that joins the focus and any point on the parabola is a focal radius. Show that if $P_1(x, y)$ is a point on the parabola defined by $y^2 = 4px$, $p > 0$, the length of the focal radius is $p + x_1$ units.

30. Find an equation for the parabola with vertex at $(1, 2)$ and focus at $(5, 2)$.

31. Find an equation for the parabola with vertex at $(p, 0)$, $p > 0$, and focus at $(0, 0)$.

4.2. Focus-directrix Definitions of the Ellipse and Hyperbola

The Ellipse

Consider the ellipse defined by $3x^2 + 4y^2 = 48$. The point P_1 with co-ordinates $(2, 3)$ is on the graph as shown in Figure 4.4, since $3(2^2) + 4(3^2) = 48$.

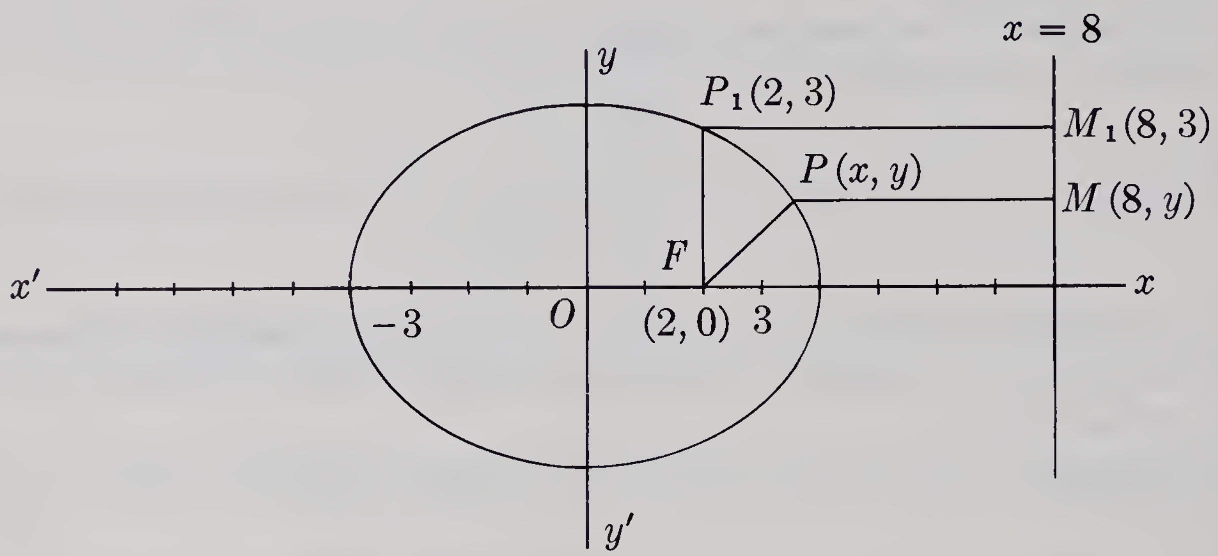


Figure 4.4

The point $F(2, 0)$ and the line whose equation is

$$x = 8$$

are also shown.

The length of the line segment P_1F is 3 units. The length of the line segment P_1M_1, where M_1 is the foot of the perpendicular from P_1 to the line defined by

$$x = 8,$$

is 6 units.
Hence,

$$\begin{aligned}\frac{P_1F}{P_1M_1} &= \tfrac{3}{6}\\ &= \tfrac{1}{2}.\end{aligned}$$

Similarly, for any point $P(x, y)$ on the graph,

$$\begin{aligned}PF &= \sqrt{(x-2)^2 + (y-0)^2}\\ &= \sqrt{x^2 - 4x + 4 + \frac{48 - 3x^2}{4}}\\ &= \sqrt{\frac{4x^2 - 16x + 16 + 48 - 3x^2}{4}}\\ &= \sqrt{\frac{x^2 - 16x + 64}{4}}\end{aligned}$$

$$= \sqrt{\frac{(x-8)^2}{2^2}}$$
$$= \left|\frac{x-8}{2}\right|$$
$$= \frac{8-x}{2}\text{, since } x \leq 4\,.$$

But $PM = 8 - x$; therefore, $\dfrac{PF}{PM} = \frac{1}{2}$.

The point F with co-ordinates $(2, 0)$ is called a focal point, or focus, of the ellipse. The line with equation $x = 8$ is called a directrix of the ellipse. The constant ratio (in this case $\frac{1}{2}$) is called the eccentricity of the ellipse. From the symmetry of the figure, we conclude that there must be a second focus and directrix to the left of the centre. This discussion suggests the following formal definition for the ellipse.

DEFINITION. An ellipse is the set of points in a plane such that the ratio of the distance of each point from a fixed point (focus) to its distance from a fixed line (directrix) is equal to a constant that is less than 1 (eccentricity).

It is customary to use $(c, 0)$ as co-ordinates for the focus and e to represent the eccentricity.

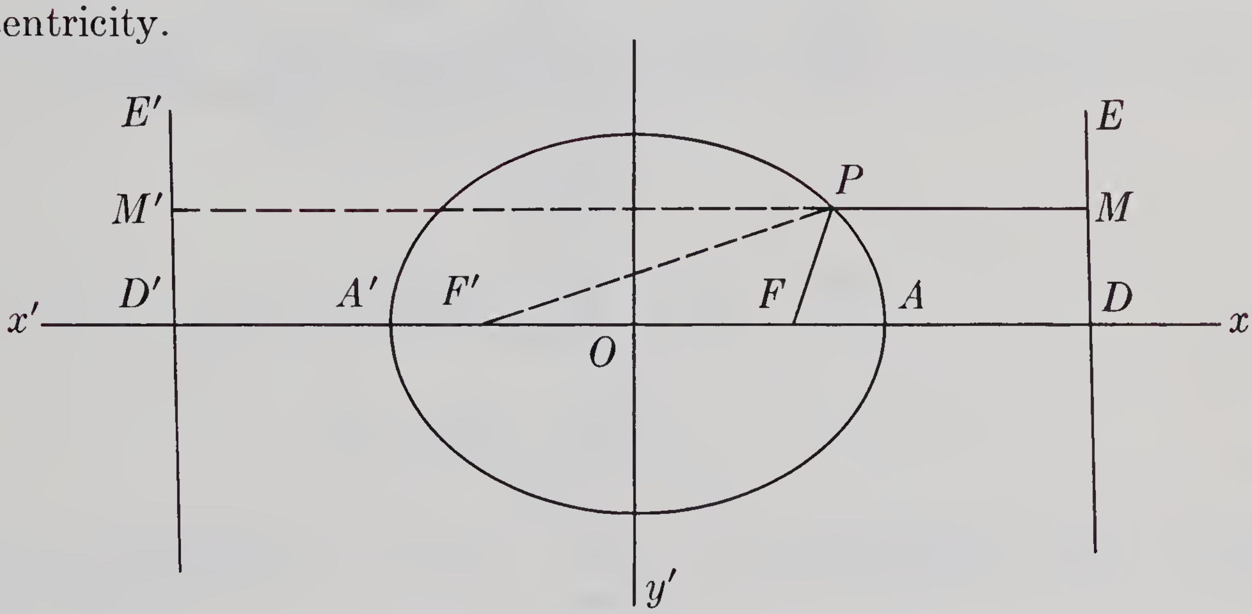


Figure 4.5

In Figure 4.5, $F(c, 0)$ and $F'(-c, 0)$ are foci of an ellipse with centre $(0, 0)$, $A(a, 0)$ and $A'(-a, 0)$ the vertices, DE and $D'E'$ the directrices. Since the vertex A lies on the ellipse,

$$\frac{FA}{AD} = e\,; \quad \text{therefore,} \quad FA = e \cdot AD\,. \qquad \text{(definition),}$$

(1)

Similarly,

$$FA' = e \cdot A'D\,. \qquad (2)$$

But

$$A'D = AD' ;$$

therefore,

$$FA' = eAD'$$

Adding (1) to (2), we obtain

$$FA + FA' = e(AD + AD') .$$

But

$$\begin{aligned} FA + FA' &= AA' \\ &= 2a , \end{aligned}$$

and

$$AD + AD' = 2OD .$$

Hence,

$$2a = e \cdot 2OD ,$$

or

$$OD = \frac{a}{e} .$$

And equation of the directrix DE is $x = \frac{a}{e}$.

Now, subtracting (1) from (2), we obtain

$$FA' - FA = e(AD' - AD) .$$

But

$$\begin{aligned} FA' - FA &= (OA' + OF) - (OA - OF) \\ &= (a + c) - (a - c) \\ &= 2c , \end{aligned}$$

and

$$\begin{aligned} AD' - AD &= AD' - A'D' \\ &= AA' \\ &= 2a . \end{aligned}$$

Hence,

$$2c = e \cdot 2a ,$$

that is,

$$c = ae ,$$

and

$$e = \frac{c}{a} .$$

The co-ordinates of the focus F are $(ae, 0)$. The co-ordinates of the focus F' are $(-ae, 0)$ and the equation of the corresponding directrix is $x = -\frac{a}{e}$.

Theorem. An equation of the ellipse with centre the origin, and foci on the x-axis at $(\pm ae, 0)$ is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1 .$$

Proof: In Figure 4.5, if $P(x, y)$ is any point on the ellipse, then

$$\frac{PF}{PM} = e,$$

or

$$PF = e \cdot PM.$$

Now

$$PF = \sqrt{(x - ae)^2 + y^2},$$

and

$$PM = \frac{a}{e} - x;$$

therefore,

$$\sqrt{(x - ae)^2 + y^2} = e\left(\frac{a}{e} - x\right),$$

$$\begin{aligned} x^2 - 2aex + a^2e^2 + y^2 &= e^2\left(\frac{a^2}{e^2} - \frac{2ax}{e} + x^2\right) \\ &= a^2 - 2aex + e^2x^2. \end{aligned}$$

Hence,

$$x^2(1 - e^2) + y^2 = a^2(1 - e^2),$$

and so

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

Conversely, if $P(x, y)$ is a point whose co-ordinates satisfy this equation, we may reverse the argument and show that

$$\frac{PF}{PM} = e.$$

Therefore, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1.$$

Now $e < 1$; therefore, $1 - e^2 > 0$. Since a^2 is also positive, $a^2(1 - e^2) > 0$.

If we write

$$a^2(1 - e^2) = b^2,$$

the equation becomes the standard form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

or

$$b^2x^2 + a^2y^2 = a^2b^2.$$

Note that

$$\begin{aligned} b^2 &= a^2(1 - e^2) \\ &= a^2 - a^2e^2 \\ &= a^2 - c^2. \end{aligned}$$

The Hyperbola

Consider the hyperbola defined by $16x^2 - 9y^2 = 144$. $F(5, 0)$ is a fixed point on the axis of symmetry and $x = \frac{9}{5}$ is the equation of a line perpendicular to the axis of symmetry.

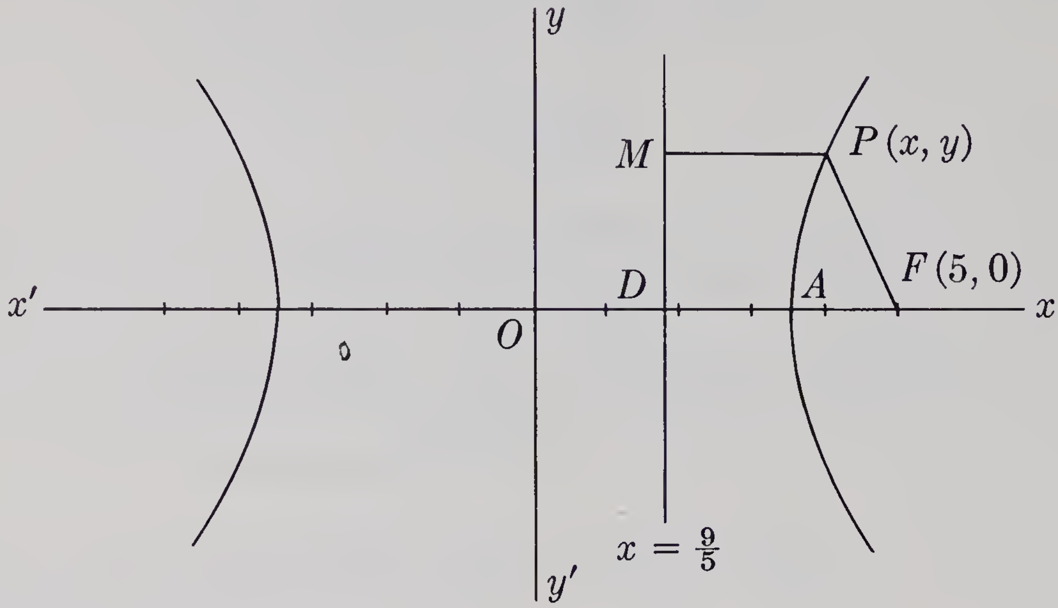


Figure 4.6

Suppose that $P(x, y)$ is any point on the hyperbola. Then, the length of the line segment PF is

$$\begin{aligned}
&\sqrt{(x-5)^2 + y^2} \\
&= \sqrt{x^2 - 10x + 25 + \frac{16x^2 - 144}{9}} \\
&= \sqrt{\frac{9x^2 - 90x + 225 + 16x^2 - 144}{9}} \\
&= \sqrt{\frac{25x^2 - 90x + 81}{9}} \\
&= \left|\frac{5x-9}{3}\right| \\
&= \frac{5x-9}{3}, \qquad \text{since } x \geq 3\,.
\end{aligned}$$

Also,

$$\begin{aligned}
PM &= x - \frac{9}{5} \\
&= \frac{5x-9}{5}\,.
\end{aligned}$$

Therefore,

$$\frac{PF}{PM} = \frac{\dfrac{5x-9}{3}}{\dfrac{5x-9}{5}} = \frac{5}{3}$$

The fixed point $F(5,0)$ is one focus of the hyperbola. The line whose equation is $x = \frac{9}{5}$ is a directrix of the curve. The constant ratio $\frac{5}{3}$ is the eccentricity of the hyperbola. Note that the eccentricity is greater than 1. This example suggests the following definition for the hyperbola.

DEFINITION. A hyperbola is the set of points in a plane such that the ratio of the distance of each point from a fixed point (focus) to its distance from a fixed line (directrix) is equal to a constant that is greater than 1 (eccentricity).

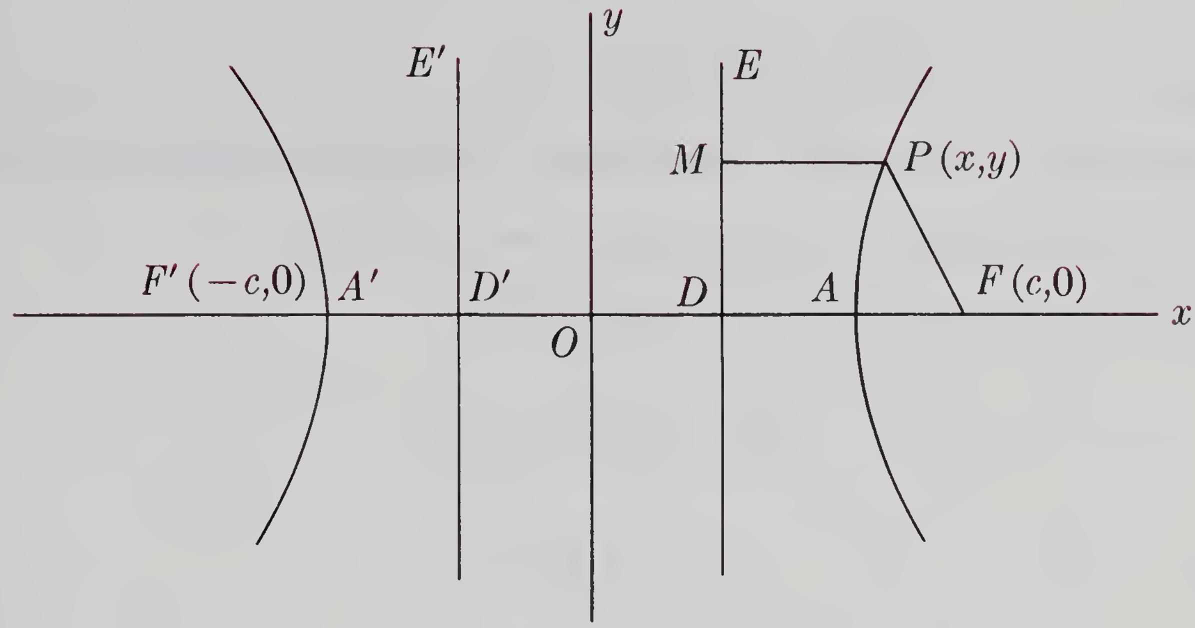


Figure 4.7

In Figure 4.7, $F(c,0)$ and $F'(-c,0)$ are the foci, $A(a,0)$ and $A'(-a,0)$ the vertices, and DE and $D'E'$ the directrices of a hyperbola with centre at the origin. By a proof anaologous to that given for the ellipse earlier in this section we may show that the co-ordinates of the foci are $(\pm ae, 0)$ and the equations of the directrices are $x = \pm\frac{a}{e}$. The proofs are required in Exercise 4.2, question (23).

Theorem. An equation of the hyperbola with centre the origin and foci on the x-axis at $(\pm ae,0)$ is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1 .$$

Proof: In Figure 4.7, if $P(x,y)$ is any point on the hyperbola,

$$\frac{PF}{PM} = e ,$$

or

$$PF = e \cdot PM .$$

Now

$$PF = \sqrt{(x-ae)^2 + y^2} ,$$

and

$$PM = x - \frac{a}{e} ;$$

therefore,

$$\sqrt{(x - ae)^2 + y^2} = e\left(x - \frac{a}{e}\right),$$

$$x^2 - 2aex + a^2e^2 + y^2 = e^2\left(x^2 - \frac{2ax}{e} + \frac{a^2}{e^2}\right)$$
$$= e^2x^2 - 2aex + a^2 .$$

Hence,

$$x^2(e^2 - 1) - y^2 = a^2(e^2 - 1) ,$$

or

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1 .$$

Conversely, if $P(x,y)$ is a point whose co-ordinates satisfy this equation, we may reverse the steps in the proof and show that $\frac{PF}{PM} = e.$

Therefore, an equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1 .$$

Now

$$e > 1;$$

hence,

$$e^2 - 1 > 0$$

and

$$a^2 > 0.$$

Therefore,

$$a^2(e^2 - 1) > 0.$$

Suppose that

$$a^2(e^2 - 1) = b^2 .$$

The equation of the hyperbola becomes

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 ,$$

or

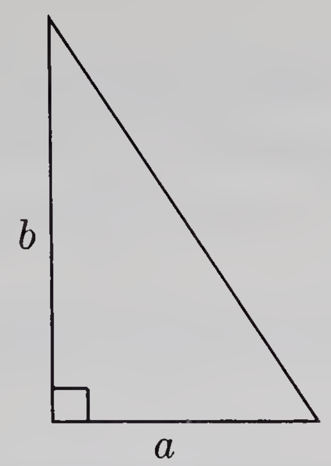
$$b^2x^2 - a^2y^2 = a^2b^2 .$$

Note that

$$b^2 = a^2(e^2 - 1)$$
$$= a^2e^2 - a^2$$
$$= c^2 - a^2 ,$$

or

$$c^2 = a^2 + b^2$$

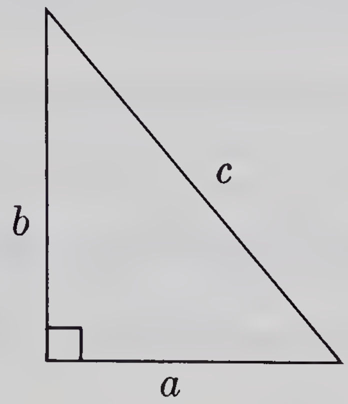


This Pythagorean relationship is useful in dealing with problems on the hyperbola. Contrast this with our earlier result on the ellipse, where

$$b^2 = a^2 - c^2\,,$$

or

$$a^2 = b^2 + c^2$$



Example 1. Find an equation of the hyperbola, centre (0,0), focus at (5,0), $e = 2$.

Solution:

$$c = 5 \text{ and } e = \frac{c}{a}\,.$$

$$\begin{aligned} a &= \frac{c}{e} \\ &= \tfrac{5}{2}\,. \\ b^2 &= c^2 - a^2 \\ &= 25 - \tfrac{25}{4} \\ &= \tfrac{75}{4}\,. \end{aligned}$$

An equation of the hyperbola is

$$\frac{x^2}{\frac{25}{4}} - \frac{y^2}{\frac{75}{4}} = 1\,,$$

or

$$12x^2 - 4y^2 = 75\,.$$

EXERCISE 4.2

For the ellipses defined by the following equations, state, (i) the co-ordinates of the foci, (ii) the equations of the directrices, (iii) the eccentricity.

1. $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$
2. $x^2 + 4y^2 = 36$
3. $\dfrac{x^2}{20} + \dfrac{y^2}{16} = 1$
4. $3x^2 + 5y^2 = 15$

For the hyperbolas defined by the following equations, state (i) the co-ordinates of the foci, (ii) the equations of the directrices, (iii) the eccentricity.

5. $\dfrac{x^2}{64} - \dfrac{y^2}{28} = 1$
6. $16x^2 - 9y^2 = 144$
7. $x^2 - 9y^2 = 9$
8. $4x^2 - 5y^2 = 20$

Find equations for the ellipses, centre (0,0), defined by the following.

9. focus at $(-5,0)$, equation of directrix $x = 20$
10. focus at $(2,0)$, $e = \frac{1}{3}$
11. vertex at $(5,0)$, $e = \frac{3}{5}$
12. vertex at $(-7,0)$, equation of directrix $x = \frac{28}{3}$

Find equations for the hyperbolas, centre (0,0), defined by the following.

13. focus at $(5,0)$, vertex at $(-3,0)$
14. vertex at $(2,0)$, equation of directrix $x = \frac{8}{5}$
15. focus at $(-13,0)$, equation of directrix $x = \frac{13}{4}$
16. focus at $(5\frac{1}{2}, 0)$, $e = \frac{11}{7}$

Find values of k so that the central conics determined by the following equations pass through the given point.

17. $4x^2 - y^2 = k$, $(8,-5)$
18. $16x^2 - ky^2 = 72$, $(-3,3)$
19. $kx^2 + 25y^2 = 100$, $(5,1)$
20. $kx^2 + 3ky^2 = 20$, $(2,-4)$
21. Find an equation for the ellipse with centre at the origin and foci on the x-axis, given that an equation of the directrix is
$$x = 10,$$
and that one focus lies on the line
$$5y = 10x - 16\,.$$

22. Find an equation for the hyperbola, centre the origin, and foci on the x-axis, given that $e = \sqrt{2}$ and that one focus lies on the line defined by

$$2y - 3x = 6\,.$$

23. Referring to Figure 4.7, prove that (i) $OD = \frac{a}{e}$, (ii) $c = ae$.

24. Find an equation for the ellipse with one focus at (2,1), one vertex at (5,1), and centre at (1,1).

25. Find an equation for the hyperbola with a focus at $(3,-4)$, the corresponding directrix given by

$$4x + 3 = 0,$$

and with both foci lying in the line

$$e = \tfrac{3}{2}. \qquad y = -4\,.$$

4.3. Alternative Definitions for the Ellipse and Hyperbola

Conics have many interesting geometrical properties other than the ones we have previously developed. In particular, there are two simple geometrical properties, one for the ellipse and one for the hyperbola, of such a fundamental nature that if a curve has one of these properties it *must* be an ellipse or hyperbola; these properties *characterize* the ellipse or hyperbola. In fact, these properties can be used as alternative definitions of the ellipse and hyperbola. First, we consider the case of the ellipse.

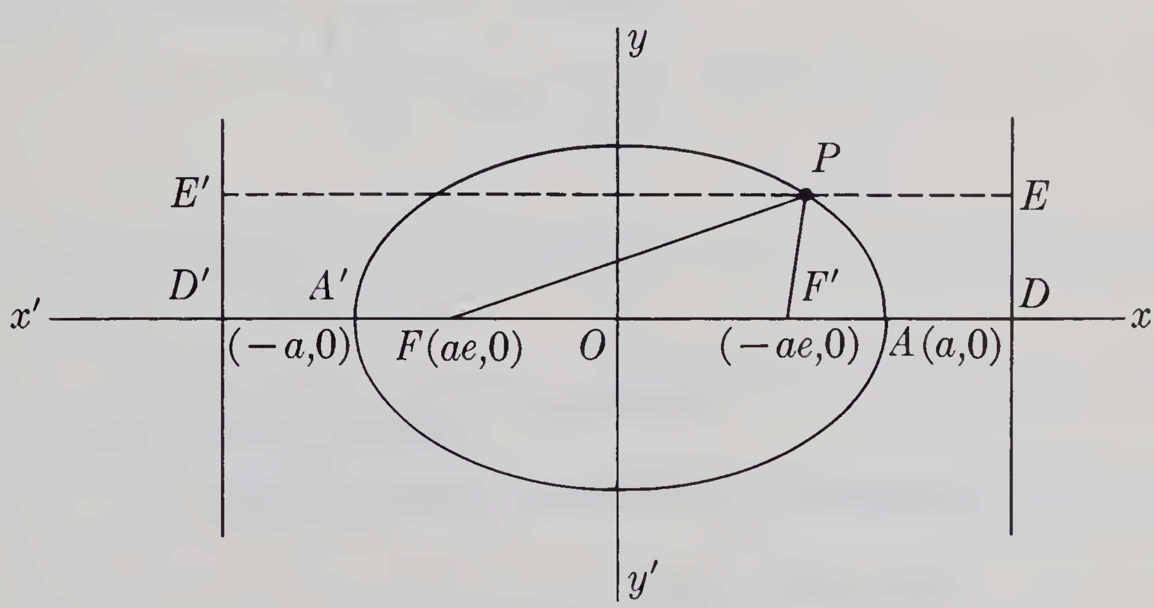


Figure 4.8

If P is any point on the ellipse in Figure 4.8, then PF and PF' are called the focal radii and

$$PF = e \cdot PE \text{ and } PF' = e \cdot PE'\,;$$

therefore,

$$
\begin{aligned}
PF + PF' &= e\cdot (PE + PE') \\
&= e\ (EE') \\
&= e\ (DD') \\
&= e\cdot 2OD \\
&= e\cdot \frac{2a}{e} \\
&= 2a\,.
\end{aligned}
$$

Note that the sum of the focal radii is constant for any point P on the ellipse and that the constant sum is equal to the length of the major axis, $2a$. This fact suggests the following alternative definition.

DEFINITION. An ellipse is a set of points such that the sum of the distances from each point to two fixed points (the foci) is a constant.

Theorem. An equation of the ellipse with centre the origin, foci on the x-axis at $(\pm c, 0)$, and with sum of the focal radii $2a\,(a > c)$, is of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

where $b^2 = a^2 - c^2$.

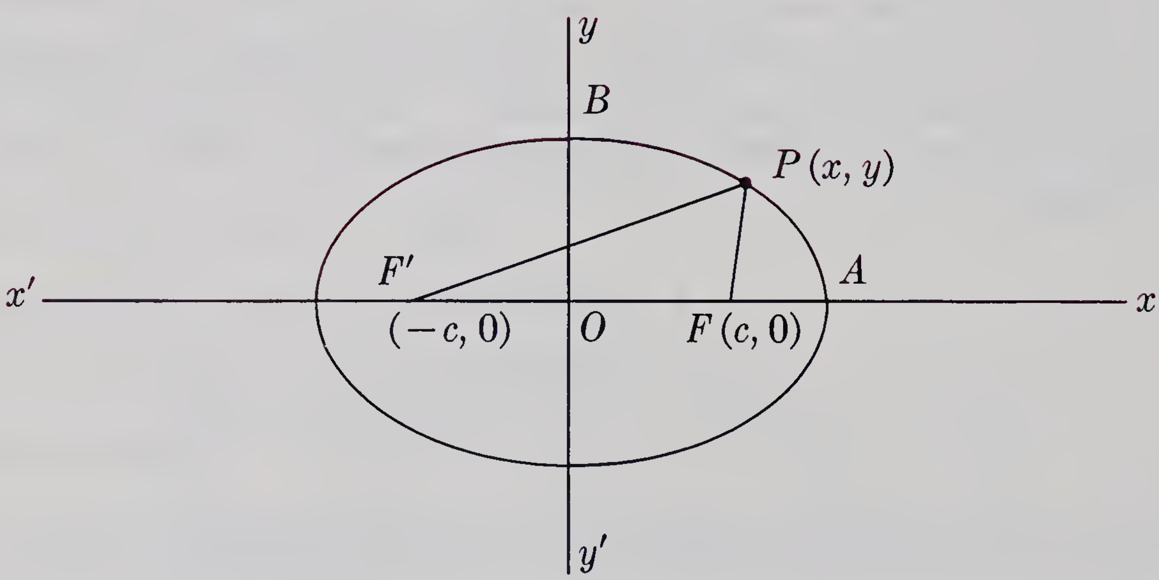


Figure 4.9

Proof: If $P(x, y)$ is any point on the ellipse (Figure 4.9), then

$$PF + PF' = 2a\,.$$

Therefore,

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a\,,$$

or

$$\sqrt{(x-c)^2 + y^2} = 2a - \sqrt{(x+c)^2 + y^2}\,.$$

Squaring both members of the equation, we find

$$x^2 - 2cx + c^2 + y^2 = 4a^2 - 4a\sqrt{(x+c)^2 + y^2} + x^2 + 2cx + c^2 + y^2\,,$$

or

$$a\sqrt{(x+c)^2 + y^2} = a^2 + cx\,.$$

On squaring again, we obtain

$$a^2(x^2 + 2cx + c^2 + y^2) = (a^2 + cx)^2\,,$$

or

$$a^2x^2 + 2a^2cx + a^2c^2 + a^2y^2 = a^4 + 2a^2cx + c^2x^2\,,$$

or

$$x^2(a^2 - c^2) + a^2y^2 = a^2(a^2 - c^2)\,.$$

But

$$\begin{aligned} a^2 - c^2 &= a^2 - (ae)^2 \\ &= a^2(1 - e^2) \\ &= b^2\,. \end{aligned}$$

Replacing $a^2 - c^2$ by b^2, we find that the equation becomes

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,,$$

Conversely, if $P(x, y)$ is any point with co-ordinates satisfying this equation, it may be shown that

$$PF + PF' = 2a\,.$$

Therefore, the equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,.$$

The major and minor axes have lengths $2a$ and $2b$ units respectively, (a and b are considered to be positive).

Example 1. Find an equation for the ellipse with one focus at $(-4, 0)$ and the semi-minor axis 6 units in length.

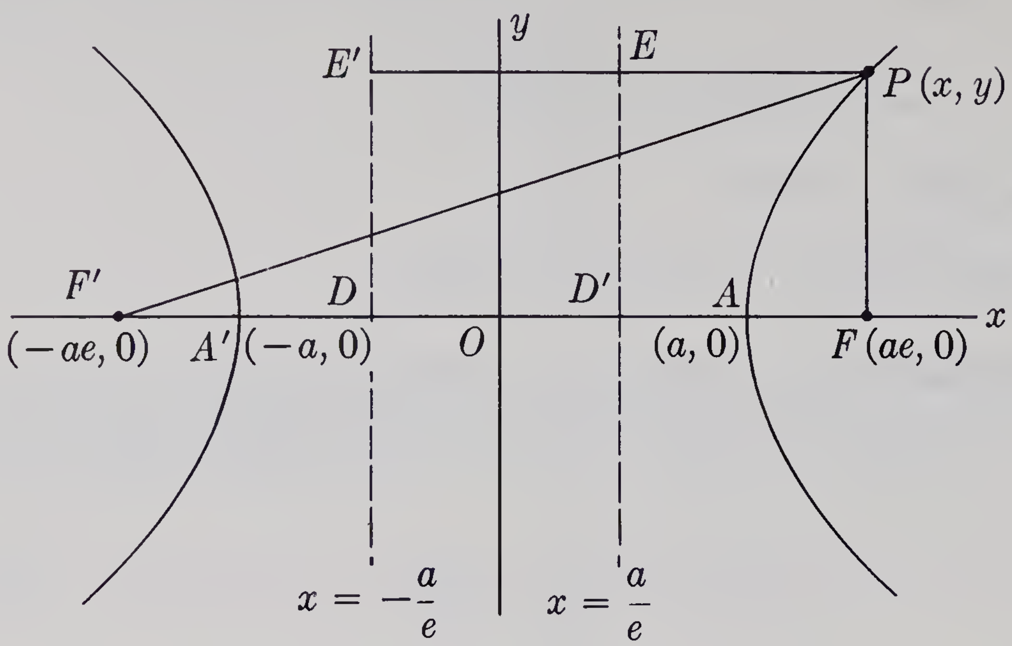
Solution:

$$c = 4\,;\; b = 6\,.$$
$$a^2 = b^2 + c^2 = 52\,.$$

An equation of the required ellipse is

$$\frac{x^2}{52} + \frac{y^2}{36} = 1\,.$$

Let us now consider the case of the hyperbola.



If P is any point on the right branch of the hyperbola in Figure 4.9, then

$$\begin{aligned} PF' - PF &= e \cdot PE' - e \cdot PF \\ &= e(PE' - PE) \\ &= e \cdot EE' \\ &= e \cdot DD' \\ &= e \cdot \frac{2a}{e} \\ &= 2a\,. \end{aligned}$$

If P is on the left branch of the hyperbola,

$$PF' - PF = -2a\,.$$

Therefore, in both cases,

$$|\,PF' - PF\,| = 2a\,.$$

The constant absolute value of the difference between the focal radii is equal to $2a$, the length of the transverse axis. The discussion above suggests the following alternative definition.

DEFINITION. A hyperbola is a set of points in a plane such that the absolute value of the difference of the distances from each point to two fixed points (the foci) is a constant.

Theorem. An equation of the hyperbola with centre the origin, foci on the x-axis at $(\pm c, 0)$, and with difference between the focal radii $2a\,(a < c)$ is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

The proof of this theorem is required in Exercise 4.3, question (17).

Example 2. For the hyperbola defined by

$$4x^2 - 9y^2 = 36 ,$$

state the values of a, b, c, and state the co-ordinates of the foci.

Solution: The defining equation may be written as

$$\frac{x^2}{9} - \frac{y^2}{4} = 1 ;$$

therefore,

$$a = 3, \quad b = 2 ,$$

and

$$\begin{aligned} c &= \sqrt{a^2 + b^2} \\ &= \sqrt{9 + 4} \\ &= \sqrt{13} . \end{aligned}$$

The foci have co-ordinates $(\pm\sqrt{13}, 0)$.

EXERCISE 4.3

1. State the standard form of the equation of the ellipse with centre $(0, 0)$, foci on the x-axis, and
   (a) $a = 5,\ b = 3$ (b) $a = 8,\ c = 6$
   (c) one focus at $(-6, 0)$, sum of focal radii $= 14$
   (d) y-intercept $= 12$, sum of focal radii $= 52$

2. State equations for the hyperbolas with centre $(0, 0)$, foci on the x-axis, and
   (a) $a = 6,\ b = 3$ (b) $a = 4,\ c = 5$
   (c) one focus at $(-10, 0)$, difference between focal radii 16
   (d) semi-conjugate axis 3 units in length, one focus at $(6, 0)$

Find equations for the central conics with centre $(0, 0)$, and foci on the x-axis, in the following cases.

3. Major axis 26 units in length and one focus at $(-12, 0)$

4. An ellipse passing through $A(7, 0)$ and $B(0, 4)$

5. An ellipse passing through $(3, 2)$ and $(1, 4)$

6. An ellipse passing through $(3, 2\sqrt{3})$ and $(\sqrt{21}, 2)$

7. Semi-transverse axis 6 units in length and one focus at $(10, 0)$

8. One vertex at $(-\sqrt{14}, 0)$, semi-conjugate axis $\sqrt{22}$ units in length

9. A hyperbola passing through $(-5, 2)$ and $(7, 10)$

10. A hyperbola passing through $(-8, 6)$ and $(4, -2)$

Find the semi-axes, foci, and eccentricity for the following:

11. $25x^2 - 9y^2 = 225$

12. $x^2 + 4y^2 = 16$

13. $x^2 - y^2 = 32$

14. $25x^2 + 169y^2 = 4225$

15. $3x^2 - 16y^2 = 48$

16. $64x^2 + 144y^2 = 1$

17. Prove that an equation of the hyperbola with centre the origin, foci on the x-axis at $(\pm c, 0)$, and with difference between the focal radii $2a\,(a < c)$ is of the form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 .$$

18. Find an equation for the ellipse, centre $(0, 0)$, foci on the x-axis, such that the length of the major axis is four times the length of the minor axis and the point $(-3, 2)$ is on the ellipse.

19. Find an equation satisfied by the co-ordinates of the points that divide the ordinates of the points on the circle defined by $x^2 + y^2 = 25$ in the ratio 3:2.

20. Find an equation satisfied by the co-ordinates of all points $P(x, y)$ such that the distance from $P(x, y)$ to $(1, 1)$ is one-half the distance from P to the y-axis.

21. Find an equation satisfied by the co-ordinates of the points that bisect the ordinates of points on the hyperbola defined by

$$16x^2 - 9y^2 = 144 .$$

22. The line segments joining $P(x, y)$ to $F(5, 0)$ and $F'(-5, 0)$ have slopes whose product is 4. Find an equation satisfied by the co-ordinates of P and graph the relation defined by this equation.

23. The focal radii definition of the hyperbola is used in range-finding to locate a hidden object. Suppose listening posts are at points F and F' that are 600 feet apart. The report of an enemy gun is heard at F' 0.1 seconds later than at F. Find an equation for the hyperbola on which the gun must be located. (Assume sound travels 1100 feet per second.) In actual practice, a second hyperbola is determined by a second pair of listening posts and the gun is located at an intersection of the two curves.

24. Show that, if a man hears the sound of a rifle firing at the same time that he hears the bullet hit its target, he must be on the branch of a hyperbola for which the target is the closer focus and the rifle is the focus farther from him.

25. In the Loran navigation system, two transmitters sending out signals replace the listening posts in (23) above. The difference in time of reception of the signals allows the navigator of a ship (or plane) to locate his vessel on a hyperbola. A second pair of signals permits him to find his exact position at an intersection of two hyperbolas. If the two transmitters are 400 miles apart and time difference in reception of the two signals indicates that a ship is 150 miles farther from one transmitter than the other, find an equation for the hyperbola on which the ship must lie.

4.4. Equations of the Conics in the Other Standard Positions

In Section 4.1 and Exercise 4.1 we developed the equation of the parabola in four standard positions. Our conclusions may be summarized as in Figure 4.10.

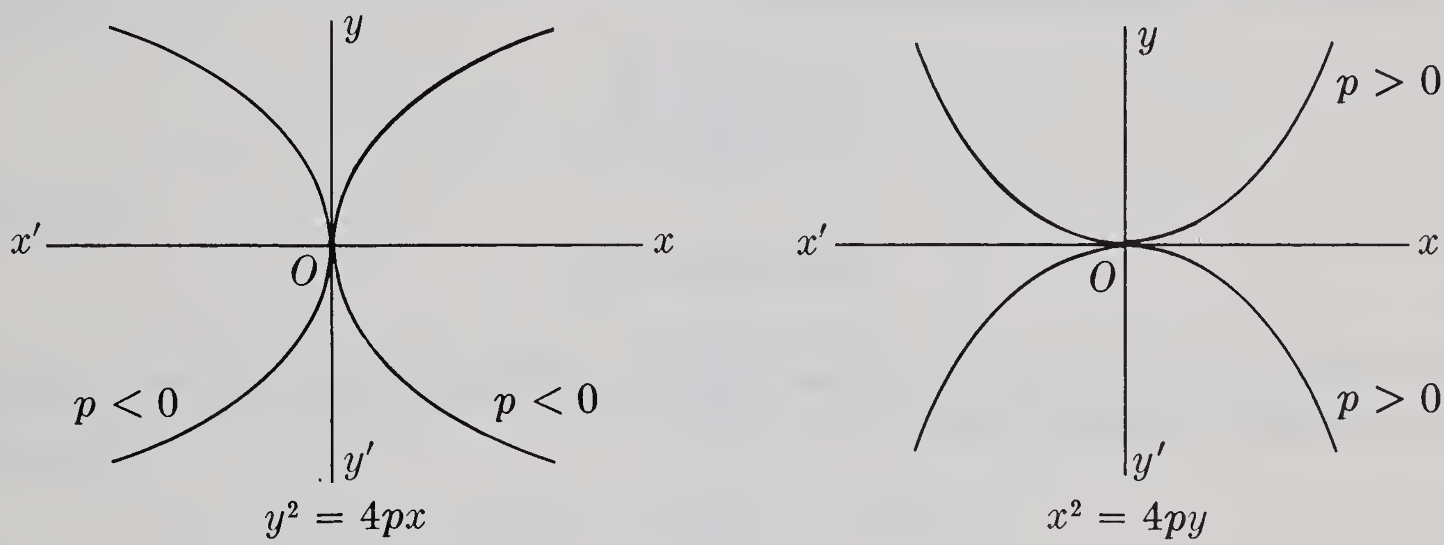


Figure 4.10

Our study of the ellipse has been restricted to the ellipse with centre at the origin and foci on the x-axis. Let us examine the corresponding ellipse with foci on the y-axis.

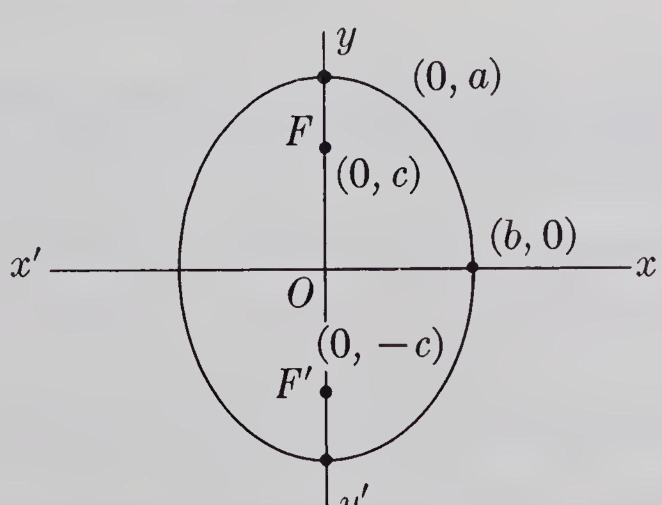


Figure 4.11

The major axis is now a segment of the y-axis and the vertices are at $(0, a)$ and $(0, -a)$. If the foci are $F(0, c)$ and $F'(0, -c)$ and $c < a$, it may be shown that the equation of the ellipse is

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1\,.$$

(The proof is required in Exercise 4.4 question (21).)

Example 1. Find an equation of the ellipse with centre $(0, 0)$, foci on the y-axis, $c = 2\sqrt{3}$, $a = 4$.

Solution:

$$\begin{aligned} b^2 &= a^2 - c^2 \\ &= 16 - 12 \\ &= 4\,. \end{aligned}$$

Therefore, an equation of the ellipse is

$$\frac{x^2}{4} + \frac{y^2}{16} = 1\,,$$

or

$$4x^2 + y^2 = 16.$$

Note that, as in the case of the ellipse with foci on the x-axis, when the equation is in standard form with foci on the y-axis,

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1\,,$$

the denominators of the left member may be associated with the squares of the x- and y-intercepts.

In considering relations that have hyperbolas as their graphs, we have also restricted our discussion to curves with foci on the x-axis. If the hyperbola has foci on the y-axis at $(0, c)$ and $(0, -c)$, and vertices at $(0, a)$ and $(0, -a)$, where $c > a$, it may be shown that the equation representing this conic is

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1\,.$$

The graph is shown in Figure 4.12 and a proof is required in Exercise 4.4, question (23).

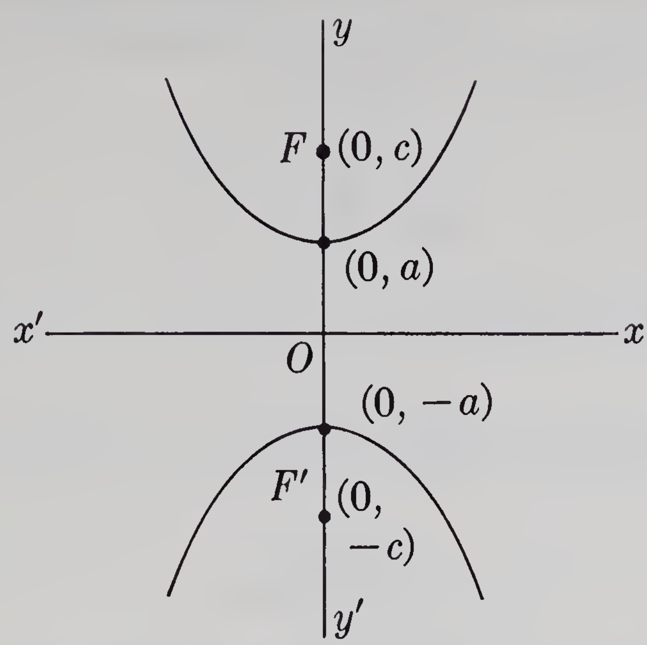


Figure 4.12

Example 2. State the values of a, b, c, e and locate the foci of the hyperbola defined by

$$\frac{x^2}{16} - \frac{y^2}{49} = -1 .$$

Solution:

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 49 + 16 \\ &= 65 . \end{aligned}$$

Therefore,

$$a = 7, \quad b = 4, \quad c = \sqrt{65} ,$$

and

$$e = \frac{c}{a} = \frac{\sqrt{65}}{7} .$$

The foci have co-ordinates $(0, \pm\sqrt{65})$.

Note that, for the hyperbola with foci on the x-axis, it is convenient to remember the equation as

$$\frac{x^2}{(\text{length of semi-axis on } x\text{-axis})^2} - \frac{y^2}{(\text{length of semi-axis on } y\text{-axis})^2} = 1 .$$

The equation of the hyperbola with foci on the y-axis may be remembered as

$$\frac{x^2}{(\text{length of semi-axis on } x\text{-axis})^2} - \frac{y^2}{(\text{length of semi-axis on } y\text{-axis})^2} = -1 .$$

Example 3. Find an equation for the hyperbola with its centre at $(0, 0)$, and foci on the y-axis if points with co-ordinates $(2, -5)$ and $(-5, 10)$ are on the graph. *Solution:* Since the equation is of the form

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$$

and $(2, -5)$ is on the graph, then

$$\frac{4}{b^2} - \frac{25}{a^2} = -1 \qquad (1)$$

and similarly,

$$\frac{25}{b^2} - \frac{100}{a^2} = -1\,, \qquad (2)$$

(1) $\times$ 4

$$\frac{16}{b^2} - \frac{100}{a^2} = -4\,, \qquad (3)$$

(2) − (3)

$$\frac{9}{b^2} = 3\,,$$

$$b^2 = 3\,.$$

Replacing b^2 by 3 in (1), we obtain

$$\tfrac{4}{3} - \frac{25}{a^2} = -1\,,$$

$$\frac{25}{a^2} = \tfrac{7}{3}\,,$$

$$a^2 = \tfrac{75}{7}\,.$$

The equation is

$$\frac{x^2}{3} - \frac{y^2}{\frac{75}{7}} = -1\,,$$

or

$$25x^2 - 7y^2 = -75\,.$$

The equilateral hyperbola

$$x^2 - y^2 = a^2\,,$$

a special case of the hyperbola with foci on the x-axis, has been discussed in Section 4 of Chapter 3. We may extend our discussion to

$$x^2 - y^2 = -a^2\,,$$

the equilateral hyperbola with foci on the y-axis.

Example 4. State the lengths of semi-axes, eccentricity, and co-ordinates of vertices and foci, for the hyperbola defined by

$$x^2 = y^2 - 25\,.$$

Solution: Since

$$x^2 = y^2 - 25\,,$$

$$\frac{x^2}{25} - \frac{y^2}{25} = -1\,.$$

The length of each semi-axis is 5 units. The co-ordinates of the vertices are $(0, \pm 5)$.

$$\begin{aligned} c^2 &= a^2 + b^2 \\ &= 50\,. \end{aligned}$$

Therefore, $c = \sqrt{50}$. The eccentricity $e = \frac{\sqrt{50}}{5}$, that is, $\sqrt{2}$, and the co-ordinates of the foci are $(0, \pm 5\sqrt{2})$.

EXERCISE 4.4

For the ellipses defined by the following equations, find (i) the co-ordinates of the vertices, (ii) the foci, (iii) the eccentricity, and (iv) sketch the curve.

1. $25x^2 + 4y^2 = 100$
2. $9x^2 + y^2 = 9$
3. $4x^2 + 3y^2 = 12$
4. $81x^2 + 16y^2 = 1$

Find an equation for the ellipse with centre at the origin, given the following:

5. Foci $(0, \pm 4)$, $a = 5$
6. Foci $(0, \pm 6)$, $e = \frac{1}{3}$
7. Vertices $(0, \pm 5)$, $e = \frac{1}{5}$
8. Foci $(\pm 15, 0)$, $a = 17$
9. Find an equation for the ellipse with centre $(0, 0)$ and foci on the x-axis that passes through $(5, -2)$ and $(2, 4)$.
10. Find an equation for the ellipse with centre $(0, 0)$ and foci on the y-axis which passes through $(1, -4)$ and $(-3, 2)$.

For the hyperbolas defined by the following equations, find (i) the co-ordinates of the vertices, (ii) the foci, (iii) the eccentricity, and (iv) sketch the curve.

11. $25x^2 - 9y^2 = -225$
12. $9y^2 - 4x^2 = 36$
13. $x^2 - y^2 = -4$
14. $3y^2 - x^2 = 9$

Find an equation for the hyperbola with centre the origin, given the following:

15. Foci $(0, \pm 5)$, length of conjugate axis 8 units
16. Vertices $(0, \pm 4)$, $e = \frac{3}{2}$

17. Foci $(0, \pm\sqrt{34})$, semi-transverse axis $3\sqrt{2}$ units

18. Foci $(0, \pm 2\sqrt{6})$, hyperbola equilateral

19. Find an equation for the hyperbola with centre $(0, 0)$ and foci on the x-axis which passes through $(5, -2)$ and $(7, 10)$.

20. Find an equation for the hyperbola with centre $(0, 0)$ and foci on the y-axis that passes through $(4, -3)$ and $(\frac{1}{4}, \sqrt{2})$.

21. Prove that an equation of the ellipse with centre the origin, foci at $(0, \pm c)$, and sum $2a\,(a > c)$ of the focal radii of any point on the curve has the form

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1\,.$$

22. Develop the equation in question (21) using the fact that the ratio of the distance from $P(x,y)$ on the ellipse to the focus $F(0, ae)$ to the distance from P to the line $y = \dfrac{a}{e}$ is equal to a constant $e(e<1)$.

23. Prove that an equation of the hyperbola with centre the origin, foci at $(0, \pm c)$, and sum $2a\,(a < c)$ the focal radii of any point has the form

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1\,.$$

24. Develop the equation in question (23), using the alternative focus-directrix definition.

25. Find an equation satisfied by the co-ordinates of all points that are twice as far from the line

$$y = 8$$

as they are from the point $(0, 2)$.

26. Find an equation for the hyperbola, centre $(0, 0)$, with one focus at $(0, -10)$ and the equation of the corresponding directrix

$$9y + 40 = 0\,.$$

27. Prove that the eccentricity of an equilateral hyperbola is $\sqrt{2}$.

28. Prove that the length of the line segment joining any point on the equilateral hyperbola defined by $x^2 - y^2 = a^2$ to its centre is a mean proportional between the focal radii drawn to that point.

29. The arch of a bridge is in the form of a segment of an equilateral hyperbola. The arch is 24 feet high, its base is 72 feet wide. Find the height of the arch at a point 8 feet from the midpoint of the base.

4.5. Intersections of Lines with Conics

If a line intersects a conic, we may find the co-ordinates of the points of intersection by solving the linear-quadratic system formed by the defining equations of the line and the conic.

Example 1. Find the co-ordinates of the points of intersection of the line defined by $5x - y - 20 = 0$ and the parabola defined by $y^2 = 50x$.

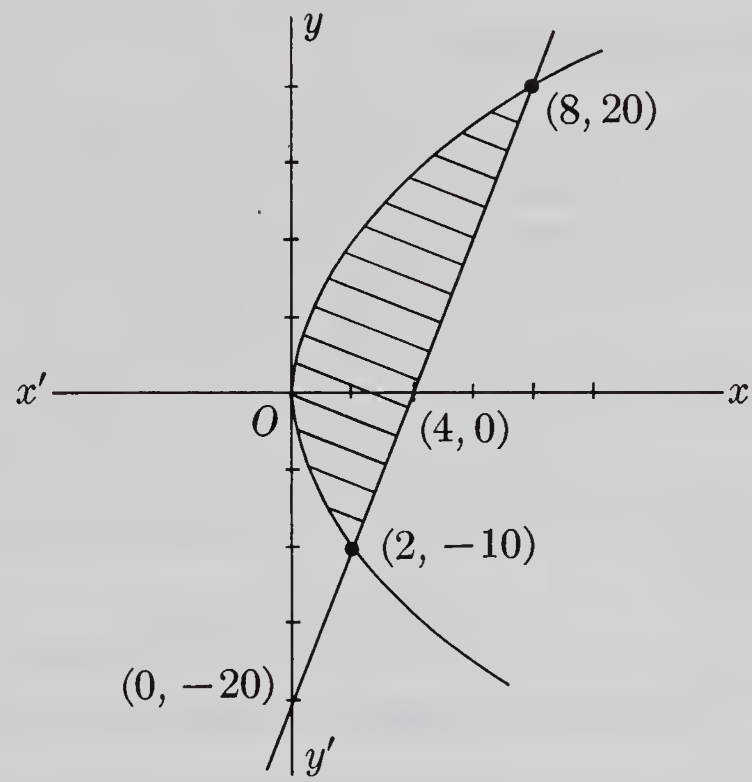


Figure 4.13

Solution: Let a point of intersection be $P(x, y)$. As P lies on both the line and the parabola, its co-ordinates satisfy both their equations. From the equation of the line, we obtain

$$5x - y - 20 = 0\,,$$
$$50x = 10y + 200\,,$$

Substituting for x in the equation of the parabola, we obtain an equation for the ordinate of a point of intersection.

$$y^2 = 10y + 200\,,$$
$$y^2 - 10y - 200 = 0\,,$$
$$(y - 20)(y + 10) = 0\,;$$

hence,

$$y = 20 \text{ or } -10\,.$$

Now

$$x = \frac{y + 20}{5}\,.$$

Therefore,

$$\begin{aligned} x &= \frac{20+20}{5} \\ &= 8, \end{aligned}$$

or

$$\begin{aligned} x &= \frac{-10+20}{5} \\ &= 2. \end{aligned}$$

The co-ordinates of the points of intersection are $(8, 20)$ and $(2, -10)$. These points are shown in Figure 4.13.

Example 2. Sketch the graph of

$$\{(x, y) \mid 5x - y - 20 \leq 0 \text{ and } y^2 \leq 50x, \quad x, y \in Re\}.$$

Solution: Consider the inequality

$$5x - y - 20 \leq 0.$$

At the origin,

$$5x - y - 20 = 0 - 0 - 20 < 0.$$

Hence, the inequality is satisfied in the closed half-plane containing the origin and bounded by the line

$$5x - y - 20 = 0.$$

(The reader will recall that a region is *closed* if its boundary is included in the region.)

The point (1, 1) satisfies the inequality $y^2 \leq 50x$.

Hence, this inequality is satisfied by the closed region containing the point (1, 1) and bounded by the parabola

$$y^2 = 50x.$$

The set of points satisfying both inequalities is the intersection of the regions found in the two preceding paragraphs (Figure 4.13).

Example 3. Sketch the graph of

$$\{(x, y) \mid 4x^2 + 9y^2 > 36 \text{ and } 16x^2 + y^2 < 16, \quad x, y \in Re\}.$$

Solution: Equations associated with these inequalities are

$$\frac{x^2}{9} + \frac{y^2}{4} = 1, \tag{1}$$

and

$$\frac{x^2}{1} + \frac{y^2}{16} = 1. \tag{2}$$

Figure 4.14 illustrates the ellipses defined by (1) and (2).

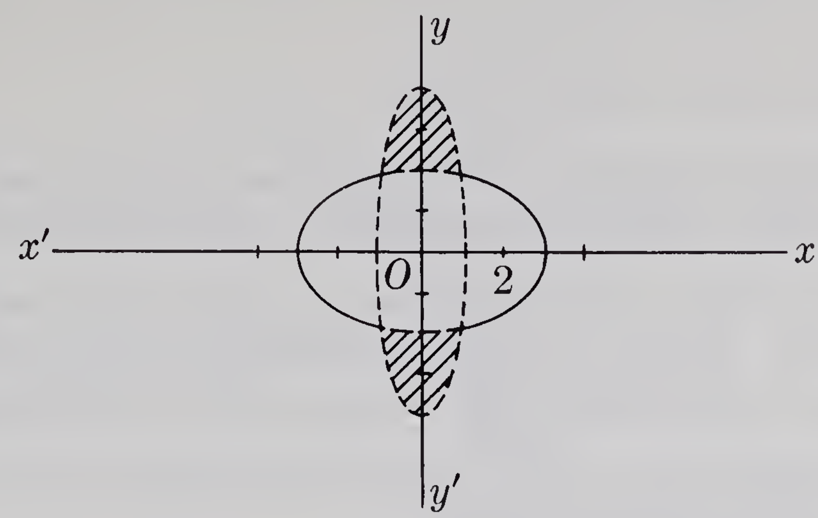


Figure 4.14

The required region is made up of the two areas shaded in Figure 4.14. The boundaries are not included. These areas contain all the points which are outside the ellipse defined by (1) and inside the ellipse defined by (2).

Note: The origin is a convenient "test point" for the inequalities in Example 3 and many other examples.

EXERCISE 4.5

Find the co-ordinates of the points of intersection of the graphs defined by the following equations.

1. $y^2 = 4x,\ x^2 = 4y$
2. $y^2 = 8x,\ 3y = 8 - 4x$
3. $x^2 + 4y^2 = 40,\ x + 2y = 8$
4. $4x^2 + 9y^2 = 36,\ 3x^2 + y^2 = 9$
5. $16x^2 + 25y^2 = 400,\ x^2 = -\frac{45}{16}y$
6. $16x^2 - 5y^2 = 80,\ 16x^2 - 25y^2 = 400.$
7. $3x^2 - y^2 = 2,\ x + y - 2 = 0$
8. $5x^2 - 4y^2 = -24,\ y^2 = 8x$

Use the solutions for questions (1) to (8) to sketch the graphs of the following $(x, y \in Re)$.

9. $\{(x, y) \mid y^2 < 4x,\ x^2 < 4y\}$
10. $\{(x, y) \mid y^2 < 8x,\ 3y > 8 - 4x\}$
11. $\{(x, y) \mid x^2 + 4y^2 < 40\} \cap \{(x, y) \mid x + 2y > 8\}$
12. $\{(x, y) \mid x^2 + 4y^2 < 40\} \cup \{(x, y) \mid x + 2y > 8\}$
13. $\{(x, y) \mid 4x^2 + 9y^2 \leq 36,\ 3x^2 + y^2 \geq 9\}$
14. $\{(x, y) \mid 16x^2 + 25y^2 \leq 400,\ x^2 \leq -\frac{45}{16}y\}$
15. $\{(x, y) \mid 16x^2 - 5y^2 \leq 80,\ 16x^2 - 25y^2 > -400\}$

16. $\{(x, y) \mid 3x^2 - y^2 < 2,\ x + y - 2 < 0,\ x > 0,\ y > 0\}$

17. $\{(x, y) \mid y^2 < 8x,\ 5x^2 - 4y^2 < -24\}$

18. $\{(x, y) \mid y^2 < 8x \text{ or } 5x^2 - 4y^2 < -24\}$

19. A square with sides parallel to the co-ordinate axes is inscribed in the ellipse $16x^2 + 9y^2 = 400$. Find the co-ordinates of its vertices and its area.

20. A line segment MN is always situated such that M is on the x-axis and N is on the y-axis. Find the equation satisfied by the co-ordinates (x, y) of point P where P is 4 inches from M and MN is 16 inches long.

21. Find the eccentricity of an ellipse if the lines joining a focus to the extremities of the minor axis are perpendicular.

22. Find an equation of the locus of the middle points of chords of an ellipse drawn from the positive end of the major axis.

23. (a) Any member of a family of chords with slope m intersects the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ at (x_1, y_1) and (x_2, y_2).
Express X and Y, the co-ordinates of the mid-point of the chord, in terms of x_1, y_1, x_2, and y_2.

(b) Use the fact that (x_1, y_1) and (x_2, y_2) are points on the ellipse to show that

$$\frac{y_2 + y_1}{x_2 + x_1} = -\frac{b^2(x_2 - x_1)}{a^2(y_2 - y_1)}$$

and hence,

$$\frac{Y}{X} = -\frac{b^2}{a^2m}$$

or

$$Y = -\frac{b^2}{a^2m}X\,.$$

(c) The line whose equation was developed in (b) is the *diameter* of the ellipse which bisects chords with slope m. Note that all diameters pass through the centre of the conic.

(d) For the ellipse with equation

$$16x^2 + 9y^2 = 144\,,$$

find an equation for the diameter bisecting chords with slope $\frac{1}{2}$.

4.6. Equations of Tangents

In Figure 4.13, the line defined by

$$5x - y - 20 = 0$$

intersects the parabola

$$y^2 = 50x$$

in two distinct points. The line (AB_1 in Figure 4.15) is a secant of the parabola.

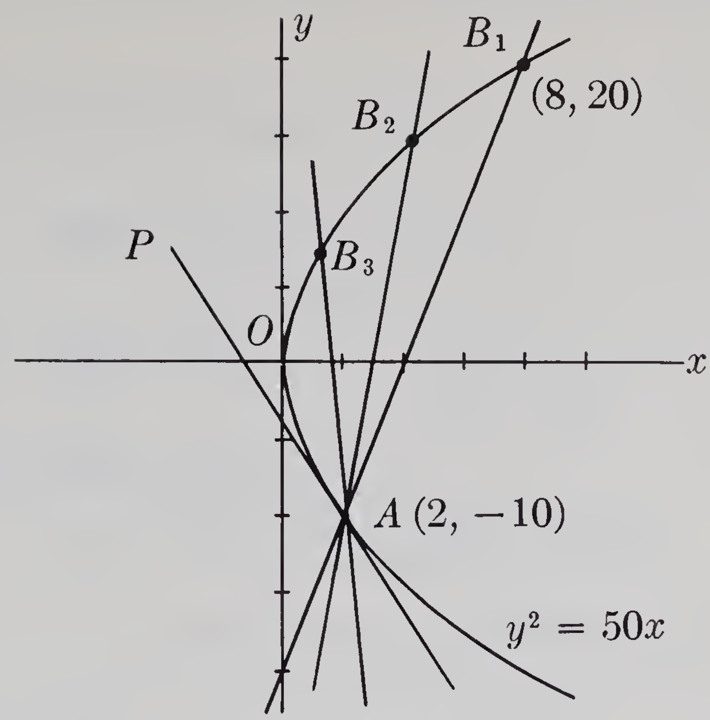


Figure 4.15

AB is the limiting position of the secant AB_n where the length of the chord AB_n approaches zero. AP is called the tangent to the parabola at point A. A similar definition may be given for the tangent at any point on a curve.

DEFINITION. The tangent to a curve at a point A on the curve is the limiting position of a secant AB_n where $\{B_n\}$ is a set of points on the curve such that the length of the chord AB_n approaches zero.

When we say that the length of AB_n approaches zero, we mean that AB_n eventually becomes *and remains* as small as we please. More formally, we say that for any $\epsilon > 0$ there exists an N such that, for all $n \geq N$, the length of AB_n is less than ϵ.

How may we obtain the equation of the tangent at $A\,(2, -10)$ in the above example? When we solved the equation $5x - y - 20 = 0$ with $y^2 = 50x$ we obtained a quadratic in y which had two distinct roots.

Referring to Figure 4.15, we see that as the length AB_n approaches zero, the two distinct roots become very close to each other in value and in the limiting case, the two roots become equal. Recall that, if the roots

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

of

$$ax^2 + bx + c = 0$$

are equal, then the discriminant $b^2 - 4ac$ must be zero. Now, in our example, we may use

$$y + 10 = m(x - 2)$$

to represent the family of lines all of which pass through $(2, -10)$. Let us solve the equation for the intersection of this line and the curve and insist on equal roots.

If

$$y + 10 = mx - 2m, \tag{1}$$

and

$$y^2 = 50x \tag{2}$$

then

$$y + 10 = \frac{my^2}{50} - 2m,$$

$$50y + 500 = my^2 - 100m,$$

$$my^2 - 50y - 100m - 500 = 0.$$

For equal roots, the discriminant is zero; that is,

$$2500 - 4(m)(-100m - 500) = 0,$$

$$25 + 4m(m + 5) = 0,$$

$$4m^2 + 20m + 25 = 0,$$

$$(2m + 5)^2 = 0;$$

therefore,

$$m = -\tfrac{5}{2}.$$

The equation of the required tangent is

$$y + 10 = -\frac{5}{2}(x - 2);$$

therefore,

$$2y + 20 = -5x + 10,$$

or

$$5x + 2y + 10 = 0.$$

The following example shows how the same method will produce the equation of a tangent to a conic if the slope is given.

Example 1. Find equations for the tangents with slope 2 to the ellipse defined by

$$x^2 + 2y^2 = 18. \tag{1}$$

Solution: The equation of the family of lines with slope 2 is

$$y = 2x + b. \tag{2}$$

Solving (1) and (2), we obtain

$$x^2 + 2(4x^2 + 4bx + b^2) = 18;$$

therefore,

$$9x^2 + 8bx + (2b^2 - 18) = 0\,.$$

For equal roots,

$$\begin{aligned} 64b^2 - 4(9)(2b^2 - 18) &= 0\,, \\ 8b^2 - 9(b^2 - 9) &= 0\,, \\ b^2 &= 81\,, \\ b &= \pm 9\,. \end{aligned}$$

The equations of the required tangents are

$$y = 2x + 9 \text{ and } y = 2x - 9\,.$$

EXERCISE 4.6

For the conic whose equation is given, find an equation for the tangent at the given point.

1. $x^2 + y^2 = 25$, $(3, -4)$
2. $x^2 = 8y$, $(4\sqrt{2}, 4)$
3. $16x^2 + y^2 = 20$, $(-1, 2)$
4. $4x^2 - y^2 = 75$, $(5, 5)$

Find equations for tangents to the given conics with the given slope.

5. $x^2 + y^2 = 4$, $m = -\frac{1}{2}$
6. $y^2 = -12x$, $m = -\frac{3}{4}$
7. $x^2 + 2y^2 = 22$, $m = \frac{1}{3}$
8. $16x^2 - 2y^2 = -1$, $m = 2$

For the conics whose equations are given, find equations for the tangents from the given external point.

9. $x^2 + y^2 = 10$, $(4, -2)$
10. $x^2 = 4y$, $(4, -5)$
11. $x^2 + 4y^2 = 4$, $(2, -1)$
12. $3x^2 - y^2 = 3$, $(3, 5)$
13. For what value of k will the line defined by $3x + y = k$ be a tangent to the curve defined by $2x^2 - y^2 = 14$?
14. Repeat question (9), representing the co-ordinates of the point of contact as (x_1, y_1), setting up two equations in x_1 and y_1, and solving these equations.
15. Repeat questions (10) to (12), using the method of question (14).
16. Prove that an equation of the tangent at (x_1, y_1) to the parabola $y^2 = kx$ is
$$y_1 y = \frac{k}{2}(x + x_1)\,.$$
17. Find the co-ordinates of the points of contact of the lines with slope $\sqrt{3}$ that are tangents to the ellipse defined by $4x^2 + 3y^2 = 5$.

18. Repeat question (16) for $x^2 = ky$.

19. Use the result of question (18) to obtain the required equation in question (2).

20. If P, Q, and R are three points on the parabola defined by $y^2 = kx$ and if the ordinates of P, Q, and R form a geometric sequence, prove that the tangents at P and R meet on a vertical line through Q.

21. (a) Find the co-ordinates of the points of contact of the tangents in question (9).

(b) Find an equation of the chord of contact in (a), that is, the chord joining the points of contact.

(c) Is it possible to obtain the equation of the chord of contact directly from the equation of the circle and the co-ordinates of the external point?

(d) State a general formula for the equation of the chord of contact for tangent drawn from (x_1, y_1) to $x^2 + y^2 = r^2$. The external point and the chord of contact are called *pole* and *polar* with respect to each other.

22. (a) Write the equation of a line passing through the point $P(2, 1)$ and having slope k. Express the condition that this line be a tangent to the parabola whose equation is

$$y = x^2$$

as a quadratic equation in k.

(b) Let the roots of the quadratic equation be k_1 and k_2. Find $k_1 + k_2$ and k_1k_2, without solving the quadratic equation.

(c) Show that the tangents from P to the parabola have points of contact $\left(\frac{k_1}{2}, \frac{k_1{}^2}{4}\right)$ and $\left(\frac{k_2}{2}, \frac{k_2{}^2}{4}\right)$

(d) Show that the equation of the chord of contact is $\dfrac{4y - k_1{}^2}{2x - k_1} = k_2 + k_1$.

Use your answer to part (b) to show that the equation of the chord of contact is

$$y = 4x - 1.$$

23. (a) Show that the chord of contact for the parabola whose equation is

$$y = ax^2$$

and the point $P(x_1, y_1)$ is given by the equation

$$y + y_1 = 2ax_1x.$$

(b) Find the point of intersection of the tangents at the intersections of the line.

$$y = 4x + 2$$

with the conic

$$y = 3x^2.$$

24. For the circle of question 21(d) and the parabola of question 23(a) show that, if the polar of P_1 passes through P_2, the polar of P_2 passes through P_1.

25. Show that the chord of contact for a parabola and a point on its directrix passes through the focus of the parabola.

26. (a) For the ellipse defined by $x^2 + 4y^2 = 25$, find an equation for the diameter that bisects chords with slope $\frac{3}{8}$.
(b) Find the co-ordinates of the end points of the diameter in (a).
(c) Show that the tangents at the end points in (b) are parallel to the original family of chords.

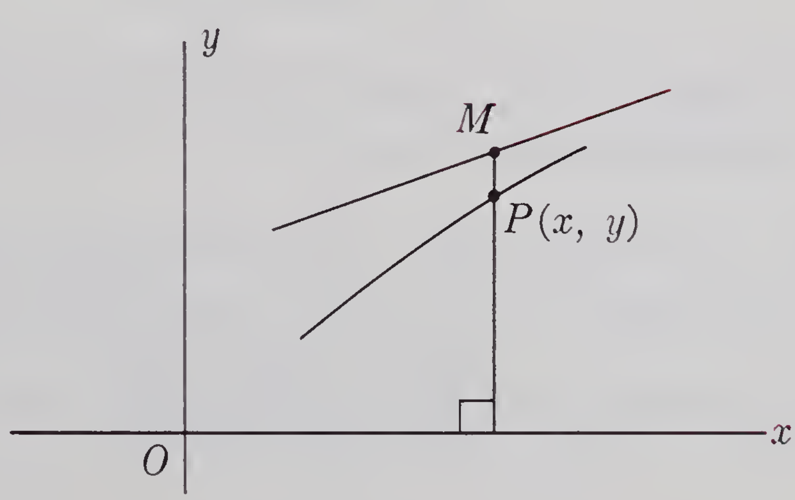
27. Let $P_1(x_1, y_1)$ lie on the conic
$$Ax^2 + 2Fxy + By^2 + 2Gx + 2Hy + K = 0\,.$$
Show that the line with equation
$$Ax_1x + F(x_1y + y_1x) + By_1y + G(x + x_1) + H(y + y_1) + K = 0$$
is a tangent to the conic at P_1.

28. The points $P(1, 2)$, $Q(1, -2)$ and $R\left(\frac{3\sqrt{2}}{4}, \sqrt{2}\right)$ lie on the conic with equation $16x^2 + y^2 = 20$. Show that, for the triangle PQR, the tangents at the vertices meet the opposite sides in collinear points.

29. The points $A(0, 0)$, $B(\frac{1}{2}, \frac{1}{2})$, $C(1, 2)$, $D(2, 8)$, $E(3, 18)$, and $F(5, 50)$ form an irregular hexagon inscribed in the conic $y = 2x^2$. Show that the points of intersection of opposite sides are collinear.

30. State the theorems suggested in the preceding two questions in general form. Is there any relationship between them?

4.7. Asymptotes of a Hyperbola (Supplementary)



DEFINITION. If $P(x, y)$ is any point on a hyperbola with foci on one of the co-ordinate axes, and the ordinate of P is produced to meet a fixed straight line at M, this straight line is called an asymptote of the hyperbola if the difference between the ordinates of P and M approaches zero as x increases indefinitely.

Example 1. Show that the line defined by $2x - 5y = 0$ is an asymptote of the hyperbola whose equation is $4x^2 - 25y^2 = 100$.

Solution: If $P(x, y)$ is any point on the hyperbola, M is the point located by producing the ordinate of P. (See Figure 4.16.) Since $4x^2 - 25y^2 = 100$, the

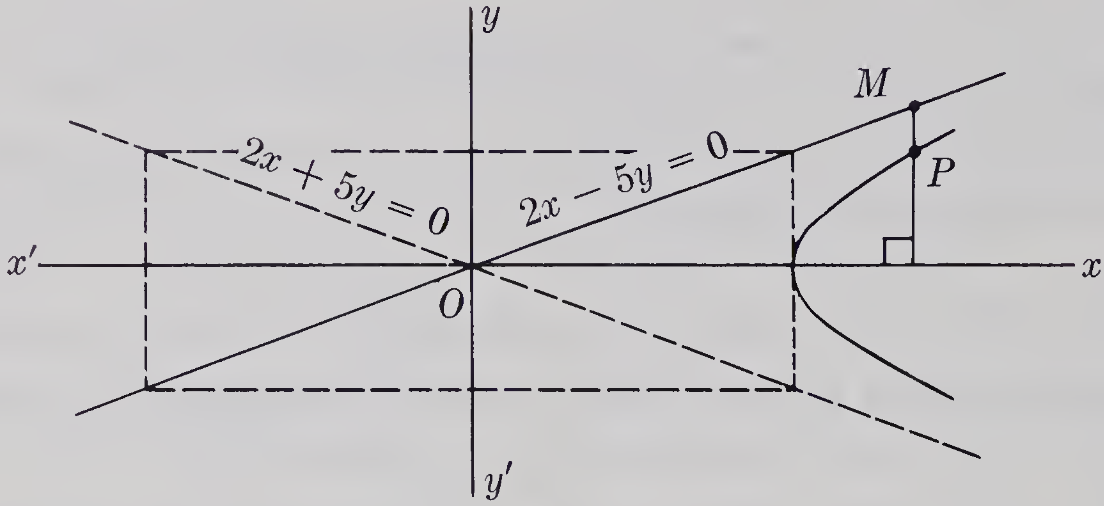


Figure 4.16

ordinate of P is given, for any value of x, by the equation

$$\begin{aligned} y &= \tfrac{1}{5}\sqrt{4x^2 - 100} \\ &= \tfrac{2}{5}\sqrt{x^2 - 25} \end{aligned}$$

The ordinate of M is $\frac{2}{5}x$ and the difference in ordinates of M and P is

$$\tfrac{2}{5}x - \tfrac{2}{5}\sqrt{x^2 - 25}$$

So that we may study the value of this expression as x increases indefinitely, we multiply it by a rational number equal to 1.

$$\begin{aligned} \tfrac{2}{5}x - \tfrac{2}{5}\sqrt{x^2 - 25} &= \left(\tfrac{2}{5}x - \tfrac{2}{5}\sqrt{x^2 - 25}\right)\frac{\frac{2}{5}x + \frac{2}{5}\sqrt{x^2 - 25}}{\frac{2}{5}x + \frac{2}{5}\sqrt{x^2 - 25}} \\ &= \frac{\frac{4}{25}x^2 - \frac{4}{25}(x^2 - 25)}{\frac{2}{5}x + \frac{2}{5}\sqrt{x^2 - 25}} \\ &= \frac{4}{\frac{2}{5}x + \frac{2}{5}\sqrt{x^2 - 25}} \end{aligned}$$

We now have a fraction with a constant numerator and with a denominator that increases indefinitely as x increases indefinitely. Therefore,

$$\frac{4}{\frac{2}{5}x + \frac{2}{5}\sqrt{x^2 - 25}}$$

approaches zero as x increases indefinitely and the difference in the ordinates of P and M also tends to zero. Thus, the line whose equation is $2x - 5y = 0$ is an asymptote of the hyperbola.

Note that, if the constant term in the equation of the hyperbola is replaced by zero, we obtain the equations of the asymptotes. If

$$4x^2 - 25y^2 = 0,$$

then

$$(2x - 5y)(2x + 5y) = 0,$$

and

$$2x - 5y = 0 \text{ or } 2x + 5y = 0.$$

For the hyperbola represented by

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

if

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

then

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{a} + \frac{y}{b}\right) = 0,$$

and

$$\frac{x}{a} - \frac{y}{b} = 0 \quad \text{or} \quad \frac{x}{a} + \frac{y}{b} = 0;$$

that is,

$$y = \frac{b}{a}x \quad \text{or} \quad y = -\frac{b}{a}x.$$

Theorem: The lines whose equations are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are the asymptotes of the hyperbola with equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

The proof of this theorem is required in Exercise 4.5, question (9).

Example 2. Find equations for the asymptotes of the hyperbolas defined by

(i) $9x^2 - 4y^2 = -1$

(ii) $x^2 - 5y^2 = 20$.

Solution:

(i) Replace the constant term by zero

If

$$9x^2 - 4y^2 = 0,$$
$$(3x + 2y)(3x - 2y) = 0.$$

Equations of the asymptotes are

$$3x + 2y = 0 \text{ and } 3x - 2y = 0.$$

(ii)

If

$$x^2 - 5y^2 = 0,$$

then

$$(x + \sqrt{5}y)(x - \sqrt{5}y) = 0.$$

Equations of the asymptotes are

$$x + \sqrt{5}y = 0 \text{ and } x - \sqrt{5}y = 0.$$

EXERCISE 4.7

State equations for the asymptotes of hyperbolas defined by each of the following equations.

1. $64x^2 - 144y^2 = 1$
2. $x^2 - y^2 = 25$
3. $49x^2 - y^2 = 16$
4. $4x^2 - 4y^2 = -45$
5. $144x^2 - 25y^2 = 3600$
6. $144x^2 - 25y^2 = -3600$
7. Sketch the hyperbolas in (5) and (6) and their asymptotes. Each hyperbola is said to be *conjugate* with respect to the other.
8. Prove that $y = 2x$ is an asymptote of the hyperbola defined by $4x^2 - y^2 = 49$.
9. Prove that $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ represent asymptotes of the hyperbola defined by $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$.
10. Find equations for the asymptotes of the hyperbola defined by $x^2 - y^2 = -81$. Compare the equations with those in question (2). What special property is possessed by the asymptotes of any equilateral hyperbola? For this reason, an equilateral hyperbola is said to be rectangular.
11. Explain how the asymptotes of a hyperbola may be used as an aid in sketching the graph of the hyperbola itself.

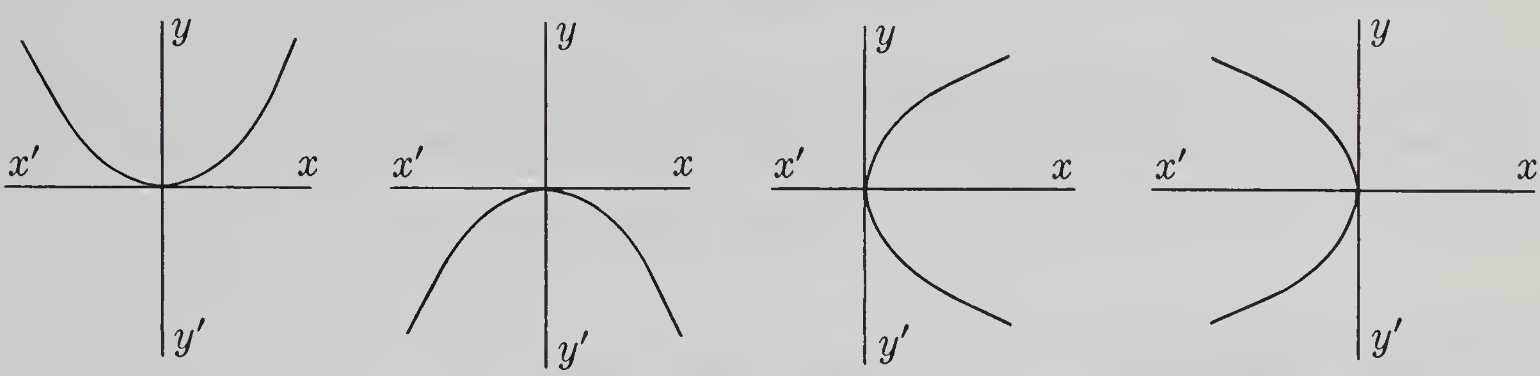
12. Prove that an asymptote of a hyperbola does not intersect the curve.

13. Sketch the hyperbola defined by $xy = 32$. Find equations for its asymptotes. State the equations of (i) the conjugate hyperbola and (ii) its asymptotes.

14. Find an equation of the hyperbola which passes through the point with coordinates $(2, -1)$ and has the line defined by $y = -\frac{4}{5}x$ as an asymptote.

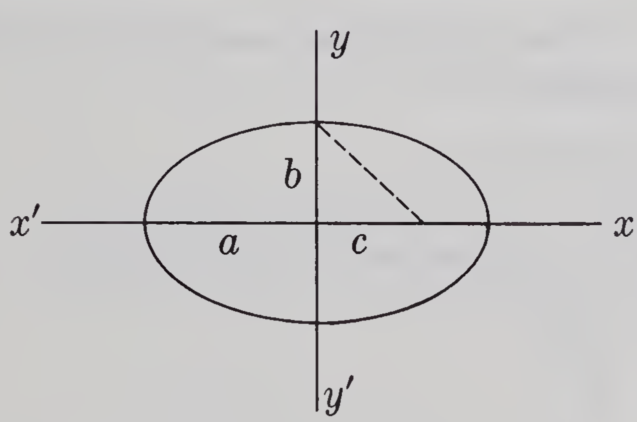
Chapter Summary

Focus-directrix definitions of parabola $(e = 1)$, ellipse $(e < 1)$, hyperbola $(e > 1)$ · Constant sum and difference definitions for the ellipse and hyperbola · Equations of the conics in the other standard positions.

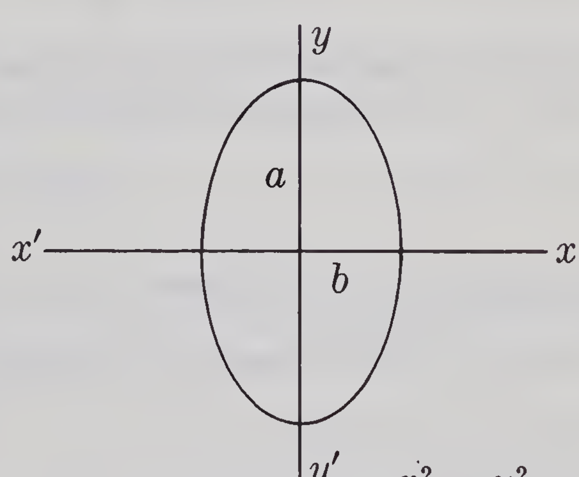


$x^2 = 4py$ $p > 0$ | $x^2 = 4py$ $p < 0$ | $y^2 = 4px$ $p > 0$ | $y^2 = 4px$ $p < 0$

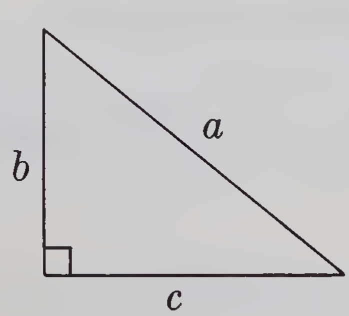
The Parabola



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

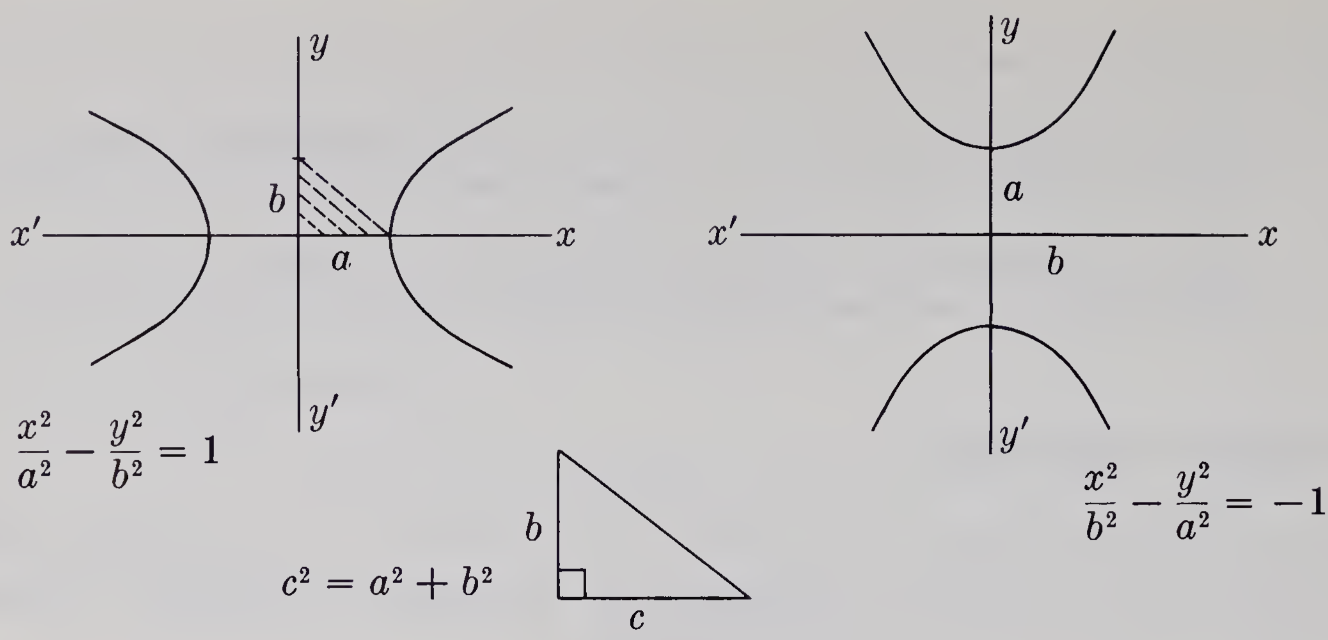


$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$$



$$a^2 = b^2 + c^2$$

The Ellipse



The Hyperbola

Asymptotes of a hyperbola (Supplementary) · Intersections of lines with conics · Intersections of conics with conics · Regions defined by inequalities · Equations of tangents to particular conics.

REVIEW EXERCISE 4

For the parabolas defined by the following equations, state (i) the co-ordinates of the focus, (ii) the equation of the directrix, (iii) the abscissa of the point whose ordinate is 2.

1. $3x^2 = 8y$ 2. $y^2 = \frac{3}{2}x$ 3. $x^2 = \frac{15}{4}y$ 4. $3y^2 = -14x$

5. Find an equation for the parabola with vertex at the origin and opening downward if the focus is on the line whose equation is
$$5x - 3y - 15 = 0.$$

Find equations for the ellipses with centres at (0,0) defined by the following.

6. Focus at (4,0), equation of directrix $x = 10$
7. Vertex at $(-10,0)$, $e = \frac{2}{5}$
8. Vertex at (5,0), equation of directrix $x = 8$

Find equations for the hyperbolas centre (0,0), defined by the following.

9. Focus at (0,5), $e = \frac{5}{3}$
10. Vertex at $(0,-8)$, equation of directrix $y = -2$
11. Vertex at $(0,\frac{3}{2})$, $e = \sqrt{2}$

Find equations for the central conics, centre (0,0), and the following.

12. Ellipse, one focus at (3,0), sum of focal radii 10
13. Hyperbola, $e = 3$, foci on y-axis, difference between focal radii 10
14. Ellipse, vertex at (0,5), equation of directrix $y = 8$
15. Hyperbola, difference between focal radii $\frac{15}{2}$, focus at $(0,-8)$
16. Hyperbola, foci on x-axis, passing through $(-4,2)$ and $(8,6)$
17. Ellipse, foci on y-axis, passing through $(\sqrt{3},-2)$ and $(1,2\sqrt{3})$
18. Equilateral hyperbola, one focus at (0,10)

Construct the graphs of the following sets of points $(x,y \in Re)$.

19. $\{(x,y) \mid x + y < 1,\ y^2 < 4x\}$
20. $\{(x,y) \mid x + y > 1 \text{ or } y^2 < 4x\}$
21. $\{(x,y) \mid x^2 + y^2 \geq 1,\ 16x^2 + 25y^2 \leq 400\}$
22. $\{(x,y) \mid x^2 < y\} \cup \{(x,y) \mid x^2 < -y\}$
23. $\{(x,y) \mid x + y > 1,\ x^2 + 25y^2 = 25,\ x > 0,\ y > 0\}$
24. For the circle defined by $x^2 + y^2 = 13$, find an equation of the tangent at (2,3).
25. For the parabola defined by $y^2 = -16x$, find an equation of the tangent with slope 2.
26. Show that the line defined by $2x - y = 8$ is tangent to the parabola represented by $x^2 = 8y$ and find the co-ordinates of the point of contact.
27. Find a value of k so that the line represented by $y = 4x + k$ will be tangent to the parabola defined by $y^2 = -20x$.
28. Find equations for the tangents from $(-4,-3)$ to the hyperbola defined by
$$\frac{x^2}{64} - \frac{y^2}{36} = 1 .$$
29. The distance between the towers of a suspension bridge is 800 feet. The lowest point on the cables is 120 feet below the tops of the towers. Assume that the form of the cables is parabolic and find an equation for the parabola.
30. Show that an equation of the tangent with slope m to the parabola with equation $y^2 = 4px$ is $y = mx + \dfrac{p}{m}$.
31. Repeat question (30) for $x^2 = 4py$.

32. Use the result of question (31) to write an equation for the tangent with slope $\frac{1}{2}$ to the parabola whose equation is $3x^2 = 8y$.

33. Find the area of the isosceles right-angled triangle inscribed in the parabola defined by $y^2 = kx$ $(k < 0)$ and having the vertex of the right angle at the origin.

34. Find an equation for the tangent common to the parabolas defined by

$$y^2 = 4x \text{ and } x^2 = 32y .$$

35. (a) Any member of a family of chords with slope m intersects the hyperbola whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. Let the mid-point of P_1P_2 be $P(X,Y)$. Express X and Y in terms of x_1, x_2, y, and y_2.

(b) Use the fact that P_1 and P_2 lie on the hyperbola to show that

$$\frac{y_1 + y_2}{x_1 + x_2} = \frac{b^2\,(x_1 - x_2)}{a^2\,(y_1 - y_2)},$$

and hence that

$$Y = \frac{b^2}{a^2 m} X .$$

(c) For the hyperbola with equation

$$3x^2 - 4y^2 = 12 ,$$

find an equation for the diameter bisecting chords with slope 10.

36. (a) The polar of $P_1\,(x_1,y_1)$ with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is the line

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1.$$

(b) Verify this fact for the ellipse

$$3x^2 + 4y^2 = 12$$

and the point (1,2).

(c) Show that the polar of a point on the diameter of the ellipse in part (a) is parallel to the family of chords that determines the diameter.

(d) Find the point of intersection of the tangents at the points of intersection of the parabola

$$x^2 + 2y^2 = 40$$

with the line

$$2y - 3x = 4 .$$

37. State equations for the asymptotes of the hyperbolas defined by each of the following equations.

(a) $25x^2 - 16y^2 = -1$ (b) $36x^2 - y^2 = 5$

(c) $x^2 - y^2 = -16$ (d) $5x^2 - y^2 = 2$

(e) $81x^2 - 169y^2 = 117$ (f) $xy = 20$

38. Find an equation for the hyperbola which passes through the point whose co-ordinates are $(3, -\frac{1}{2})$ and has the line defined by $3x + 8y = 0$ as an asymptote.

39. Let a point $P_1(x_1, y_1)$ lie on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1\,.$$

Draw a line through P_1 parallel to the y-axis to intersect the nearer asymptote at Q and the farther asymptote at Q'. Let the perpendicular from P_1 to the nearer asymptote be P_1R and to the farther asymptote be P_1R'. Let the lengths of P_1Q, P_1Q', P_1R and P_1R' be d, d', p and p' respectively. Let each asymptote make an angle θ with the x-axis.

(a) Show that

$$p = d\cos\theta$$

and that

$$p' = d'\cos\theta\,.$$

(b) Express d and d' in terms of a, b, and x.

(c) Show that

$$dd' = b^2$$

(d) Show that

$$pp' = \frac{a^2b^2}{a^2 + b^2}\,.$$

40. (a) The polar of $P_1(x_1, y_1)$ with respect to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

is the line

$$\frac{x_1x}{a^2} - \frac{y_1y}{b^2} = 1\,.$$

(b) Verify this fact for the hyperbola

$$2x^2 - 5y^2 = 20$$

and the point (2, 3).

(c) Show that an asymptote to a hyperbola cannot be a polar of any point, with respect to the hyperbola.

41. Show that the polar of a point on the diameter of a hyperbola is parallel to the system of chords that determine the diameter.

42. For the circle, parabola, ellipse, and hyperbola show that the point of intersection of the polars of two points is the pole of the line joining the two points.

Chapter 5

TRIGONOMETRIC FUNCTIONS

5.1 Definition of the Trigonometric Functions

In an earlier course, we learned that the trigonometric functions may be defined by reference to a co-ordinate system. We choose a point $P(x, y)$ on the circle with centre at the origin and radius r. A counterclockwise rotation from a fixed initial ray OX to a final ray OP determines a unique angle θ. The unit of measurement is the radian or the degree.

The measure of the angle, in either unit, is a real number, and π radians $= 180°$.

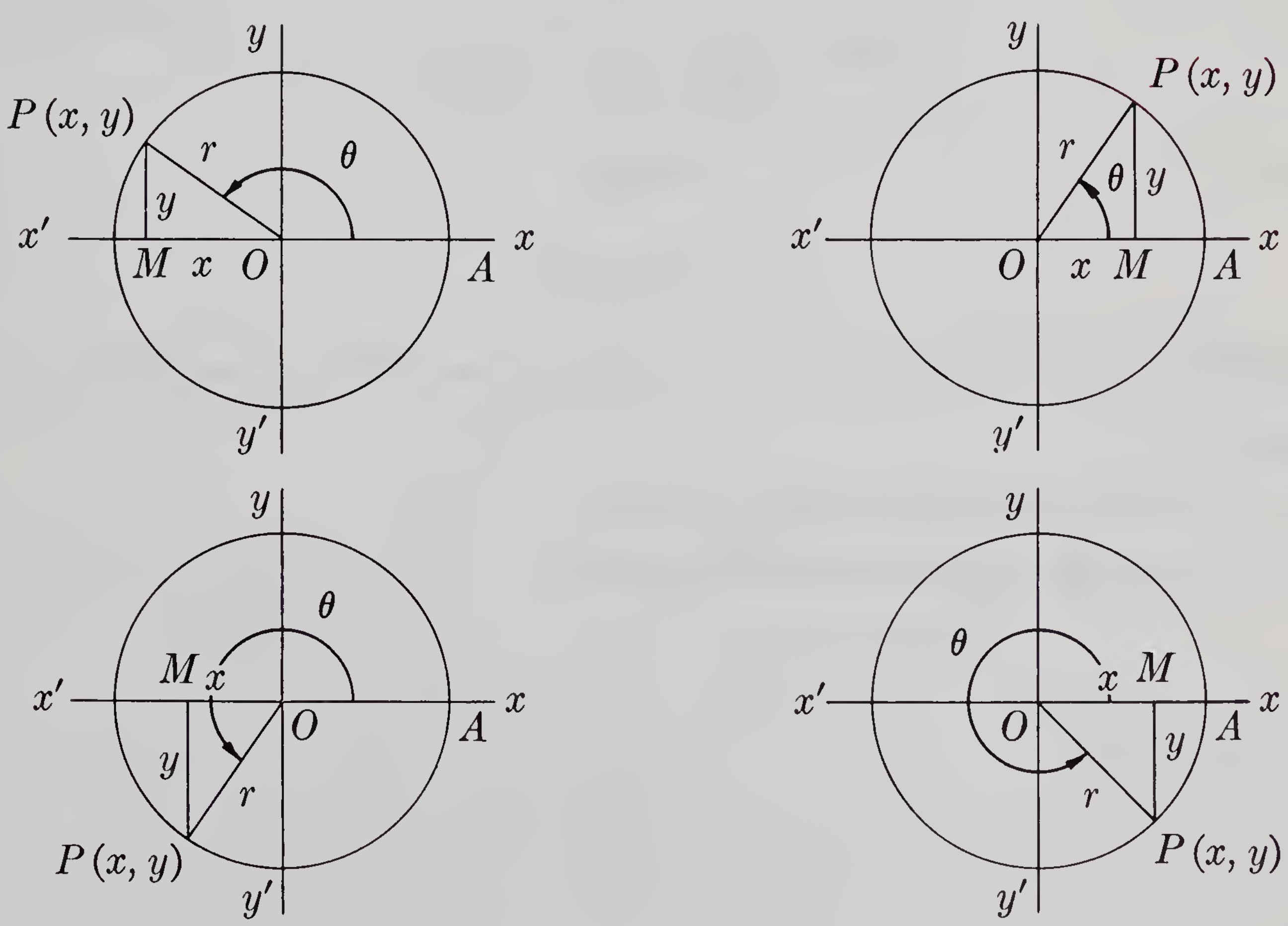


Figure 5.1

The values of the trigonometric functions (θ in radians, $0 \leq \theta \leq 2\pi$) are then defined by the following equations.

sine of $\theta = \sin \theta = \dfrac{y}{r}$, cosecant of $\theta = \csc \theta = \dfrac{r}{y}(y \neq 0)$,

cosine of $\theta = \cos \theta = \dfrac{x}{r}$, secant of $\theta = \sec \theta = \dfrac{r}{x}(x \neq 0)$,

tangent of $\theta = \tan \theta = \dfrac{y}{x}(x \neq 0)$, cotangent of $\theta = \cot \theta = \dfrac{x}{y}(y \neq 0)$.

In Section 5.2, we shall show that these are true functions defined in the usual way.

It follows from these definitions that the sine, cosine, and tangent functions (along with their reciprocal functions) are positive in the quadrants indicated in Figure 5.2 (the CAST diagram).

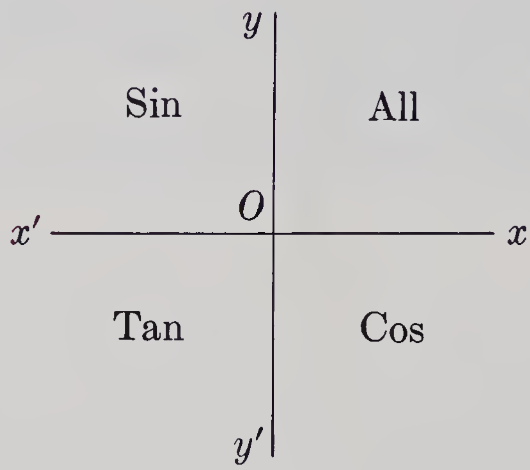


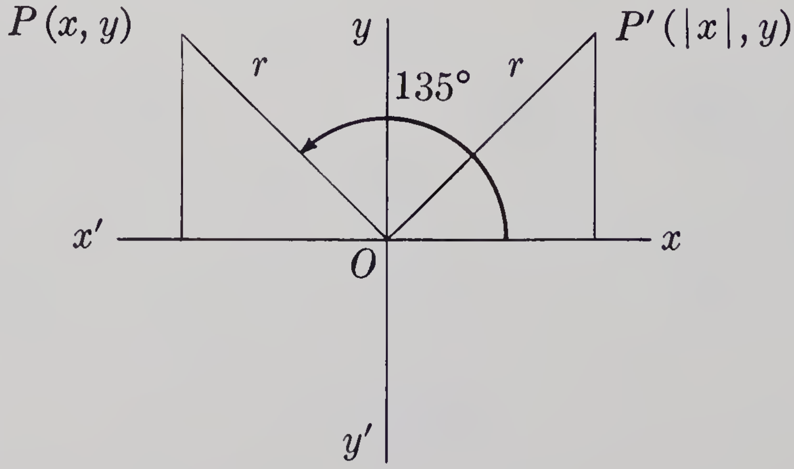
Figure 5.2

Example 1. Find the values of (a) $\cos 43°$, (b) $\sec 135°$, (c) $\tan 220°$, (d) $\sin 345°$.

Solution:

(a) From the tables, $\cos 43° \simeq .7314$.

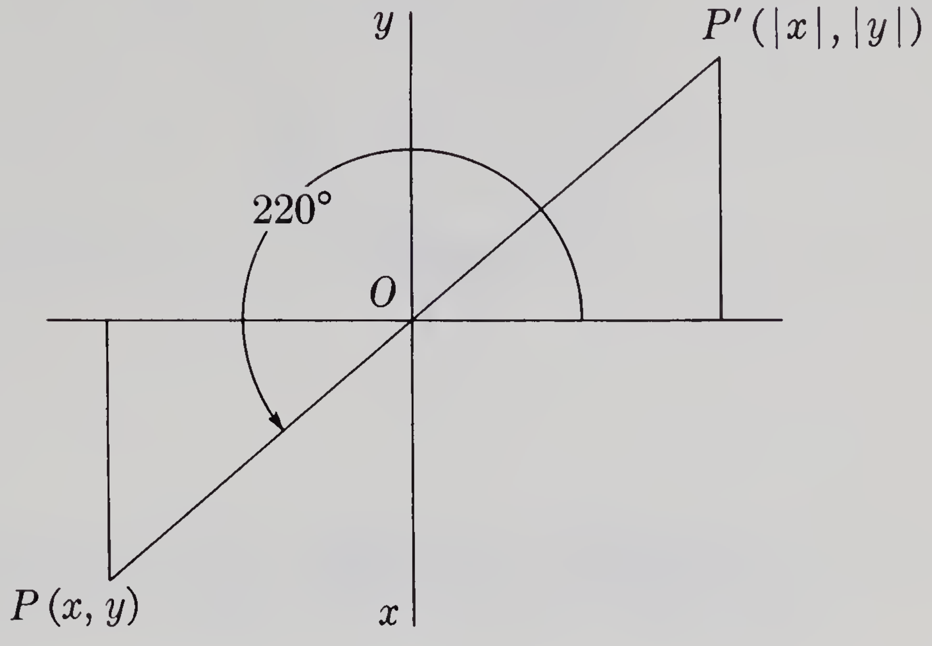
(b) Sec $135°$ is negative (CAST rule).



From the diagram,

$$\begin{aligned} \sec 135^\circ &= \sec(180^\circ - 45^\circ) \\ &= -\sec 45^\circ \\ &= \simeq -1.414\,. \qquad \text{(tables)} \end{aligned}$$

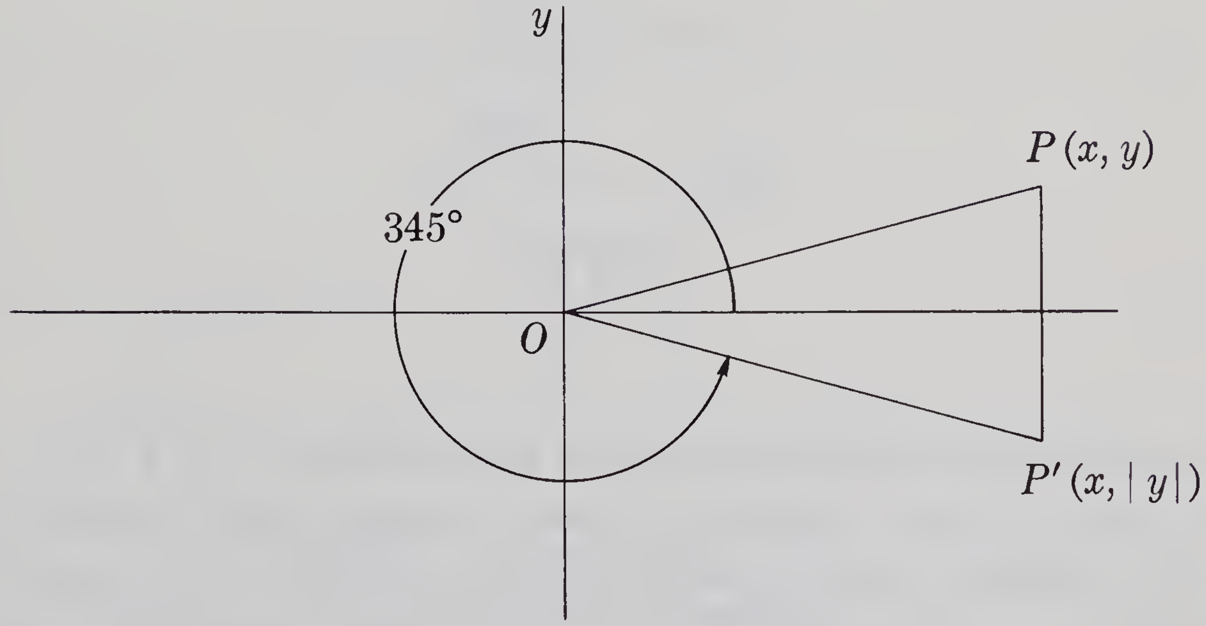
(c)



From the diagram,

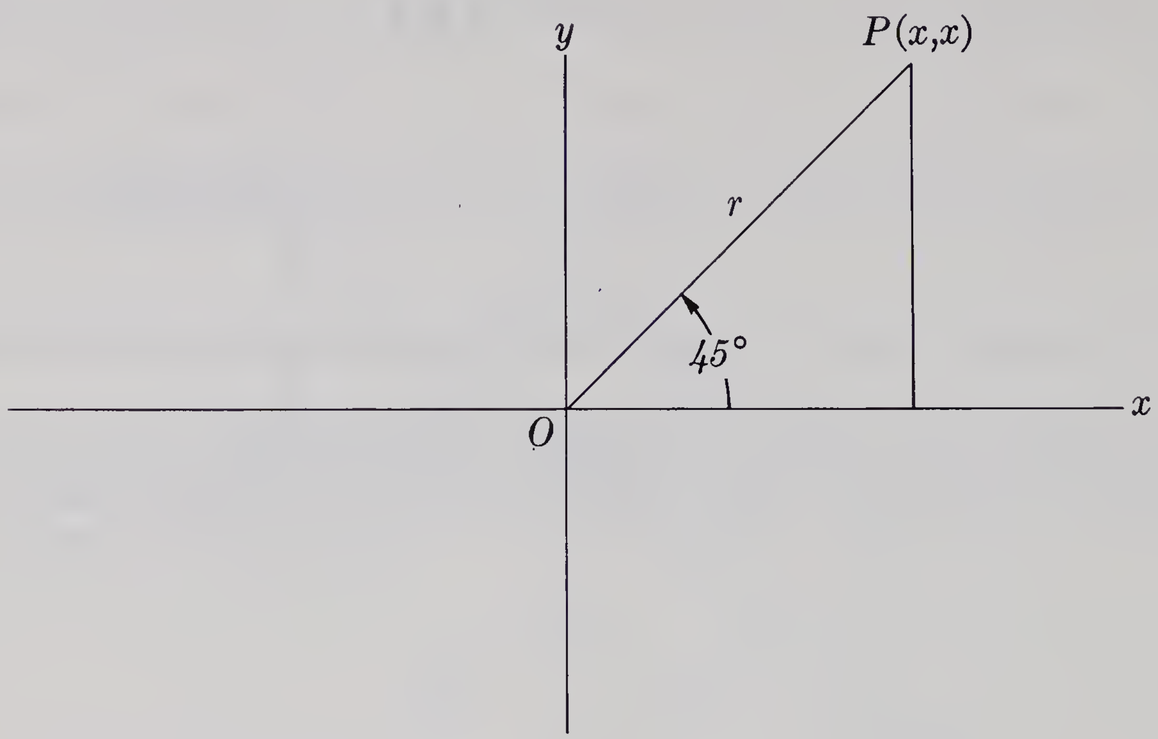
$$\begin{aligned} \tan 220^\circ &= \tan(180^\circ + 40^\circ) \\ &= \tan 40^\circ \\ &\simeq .8391\,. \qquad \text{(tables)} \end{aligned}$$

(d)



$$\begin{aligned} \sin 345^\circ &= \sin(360^\circ - 15^\circ) \\ &= -\sin 15^\circ \\ &\simeq -.2588\,. \qquad \text{(tables)} \end{aligned}$$

Example 2. Without using tables, find the values of $\cos 45°$ and $\tan \frac{\pi}{4}$.



Solution: If the point P has co-ordinates (x, x), then $OP = x\sqrt{2}$ (Pythagoras). In this isosceles right-angled triangle, angle xOP has 45 as its measure in degrees (or $\frac{\pi}{4}$ in radians). Therefore,

$$\cos 45° = \frac{x}{r}$$

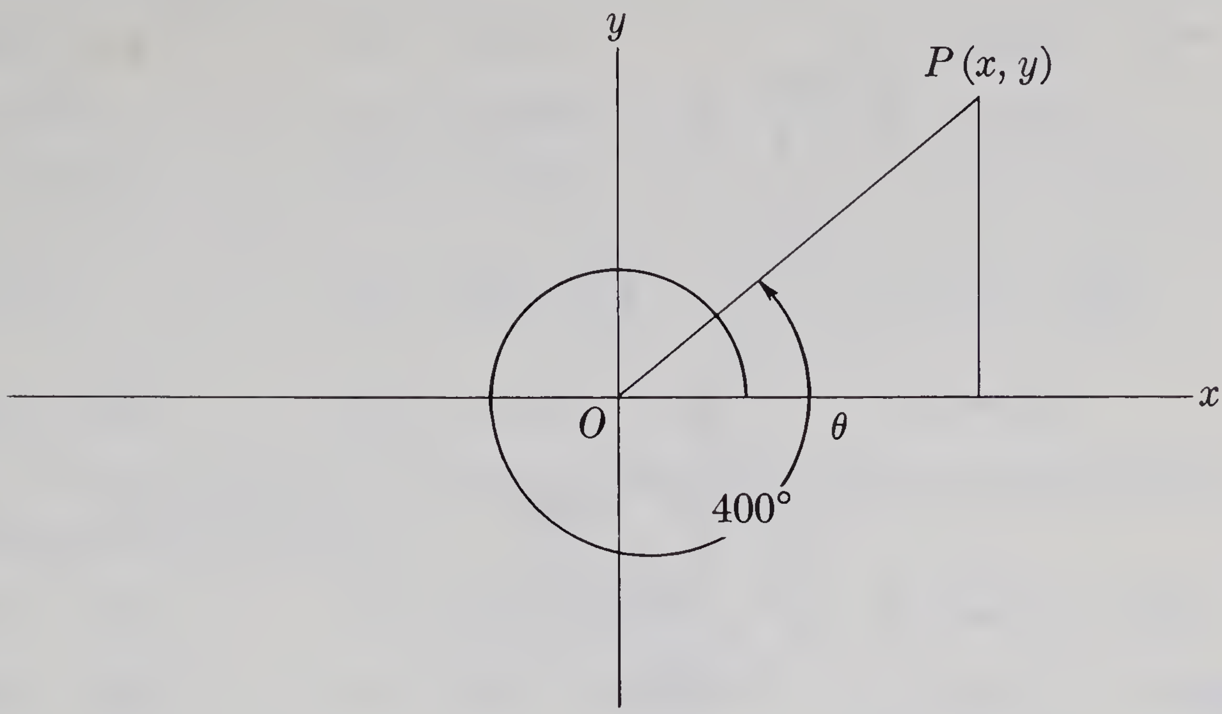
$$= \frac{1}{\sqrt{2}},$$

and

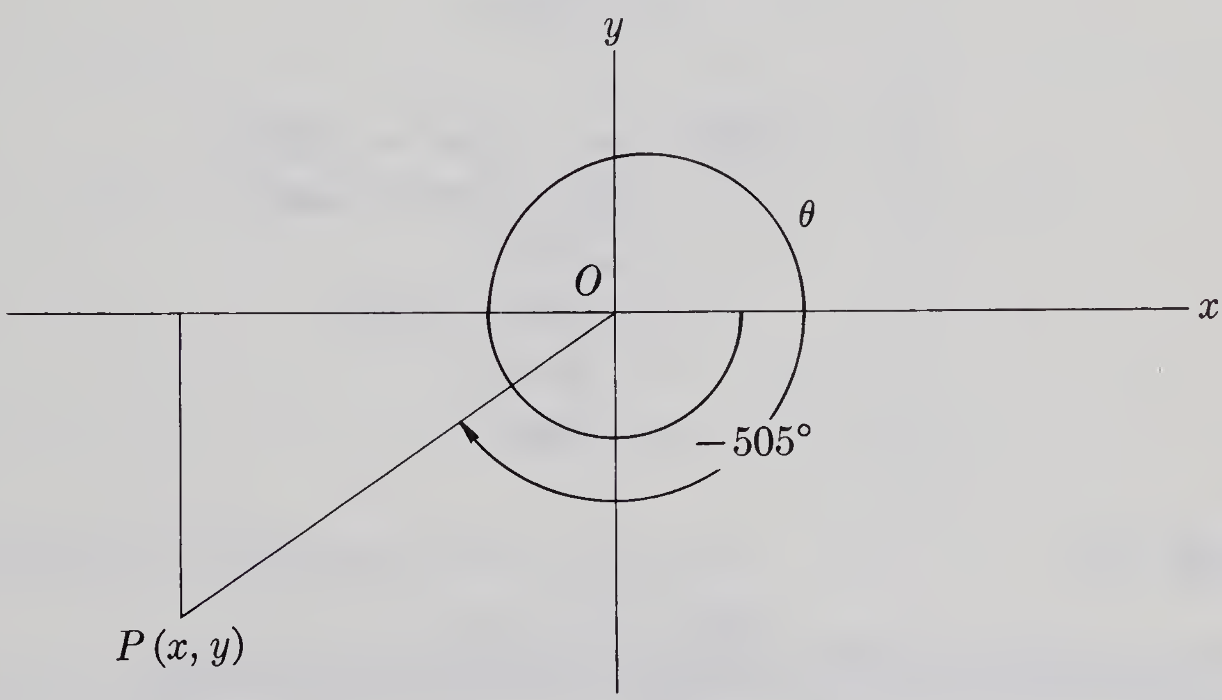
$$\tan \frac{\pi}{4} = \frac{x}{x}$$

$$= 1 .$$

Let us extend our definitions beyond those given for $0 \leq \theta \leq 2\pi$. It is important to do so because in practice we deal with angles whose measure in degrees may be 400 or 5000. Our own planet rotates through 360° in one day, 720° in two days, and so on. Any rotating object moves through angles that increase with time. The diagram shows what we mean by an angle of 400°. Our definitions of the trigonometric functions for $0 \leq \theta \leq 2\pi$ imply that the values of these functions for 400° are the same as the values for the coterminal angle 40°. The ray OP is a common coterminal ray.



Similarly, the values of the trigonometric functions of $-505°$ (a clockwise rotation) are identical with those for its coterminal angle, 215°.

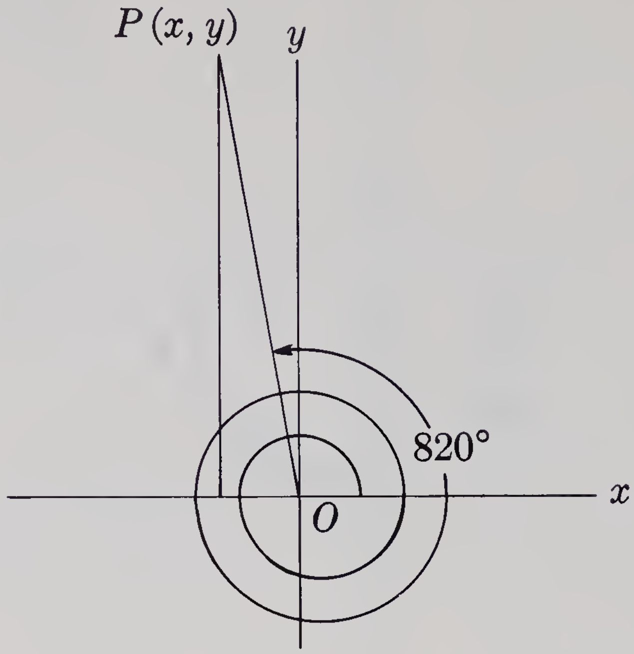


Note that

$$\begin{aligned} 215° &= -505° + 720° \\ &= -505° + 2(360)°. \end{aligned}$$

In general, to find the trigonometric functions of any angle θ, we add or subtract multiples of $2\pi(360°)$ until we obtain an angle ϕ between 0 and $2\pi(0°$ and $360°)$. Then, any trigonometric function of θ has the same value as the trigonometric function of the coterminal angle ϕ.

Example 3. Find the values of (a) $\tan 820°$, (b) $\sin(-1100°)$.



Solution:

(a)

$$\begin{aligned}\tan 820° &= \tan(720° + 100°)\\ &= \tan 100°\\ &= \tan(180° - 80°)\\ &= -\tan 80°\\ &\simeq -5.6713\,. &&\text{(tables)}\end{aligned}$$

(b)

$$\begin{aligned}\sin(-1100°) &= \sin(-1100° + 4(360°))\\ &= \sin(-1100° + 1440°)\\ &= \sin(340°)\\ &= \sin(360° - 20°)\\ &= -\sin 20°\\ &\simeq -.3420\,. &&\text{(tables)}\end{aligned}$$

EXERCISE 5.1

Use tables to find the following values of trigonometric functions.

1. $\sin 48°$	2. $\cos 100°$	3. $\tan 17°$
4. $\sin 228°$	5. $\cos 260°$	6. $\tan 343°$
7. $\tan \frac{1}{12}\pi$	8. $\sin \frac{3}{5}\pi$	9. $\cos(-\frac{1}{18}\pi)$
10. $\tan \frac{25}{12}\pi$	11. $\sin \frac{18}{5}\pi$	12. $\cos \frac{19}{18}\pi$
13. $\sin 523°$	14. $\cos 837°$	15. $\tan(-395°)$
16. $\sin(-125°)$	17. $\cos 430°$	18. $\tan 1510°$

Without using tables, find the following values.

19. $\cos 60°$ 20. $\sec 45°$ 21. $\tan 60°$

22. $\sin 120°$ 23. $\cos 225°$ 24. $\cot 300°$

25. $\csc(-60°)$ 26. $\sin 945°$ 27. $\cos(-750°)$

28. $\tan \frac{3\pi}{4}$ 29. $\sin\left(-\frac{7\pi}{3}\right)$ 30. $\csc\left(-\frac{2\pi}{3}\right)$

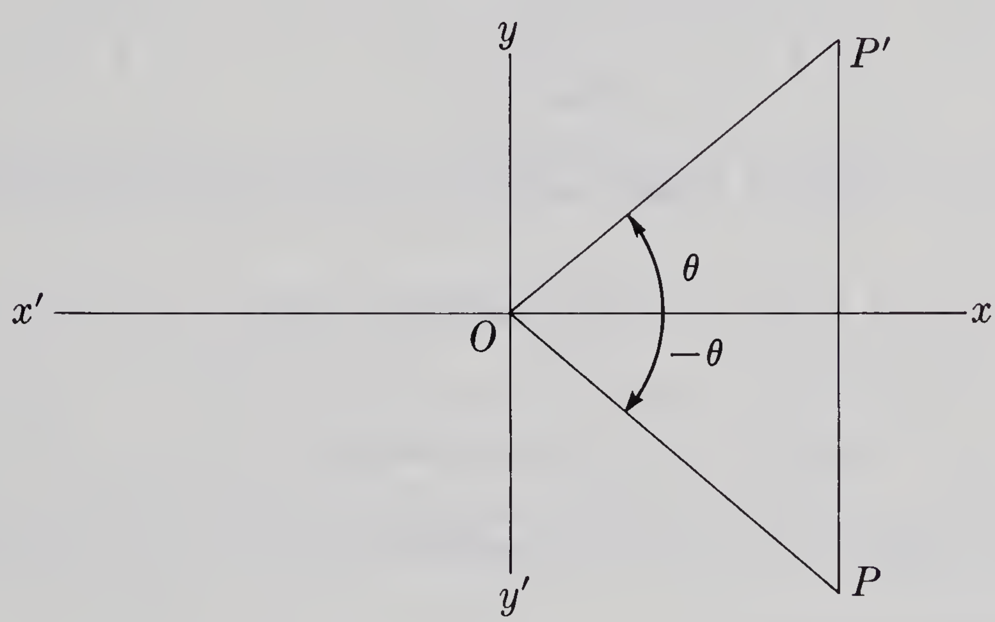
31. $\cos(-\frac{33}{4}\pi)$ 32. $\cot \frac{13}{6}\pi$ 33. $\sec \frac{17}{3}\pi$

Find, to the nearest degree, two angles θ, $0° \leq \theta \leq 360°$, for the following values of each trigonometric function.

34. $\sin\theta = .4511$ 35. $\cos\theta = .8457$ 36. $\tan\theta = 1.0724$

37. $\cos\theta = .6820$ 38. $\sin\theta = -.7641$ 39. $\tan\theta = -2.0511$

40. $\sec\theta = 1.374$ 41. $\csc\theta = 3.924$ 42. $\cot\theta = -.2155$



43. Referring to the diagram above, prove that
$$\sin(-\theta) = -\sin\theta.$$
$-\theta$ is a negative fourth quadrant angle and θ is the corresponding positive acute angle.

44. As in question (43) prove (a) $\cos(-\theta) = \cos\theta$, (b) $\tan(-\theta) = -\tan\theta$. Similar results may be obtained in all four quadrants and hence are true for any angle θ.

45. Use the results of questions (43) and (44) to find the following values.
(a) $\sin(-60°)$ (b) $\cos(-130°)$ (c) $\tan(-200°)$
(d) $\cos(-1540°)$ (e) $\tan(-30°)$ (f) $\sec(-1000°)$

5.2 Domain, Range, and Graph of the Sine and Cosine Functions

Consider the function that we have called "sine". Each angle θ determines one and only one final ray OP. The ratio $\sin\theta$ is independent of the radius of the circle but is determined by the angle θ. Thus, for each θ, there is exactly one value of $\sin\theta$. The set of ordered pairs $(\theta, \sin\theta)$ satisfies the requirement for the set to be a function, namely, that no two pairs have the same first element.

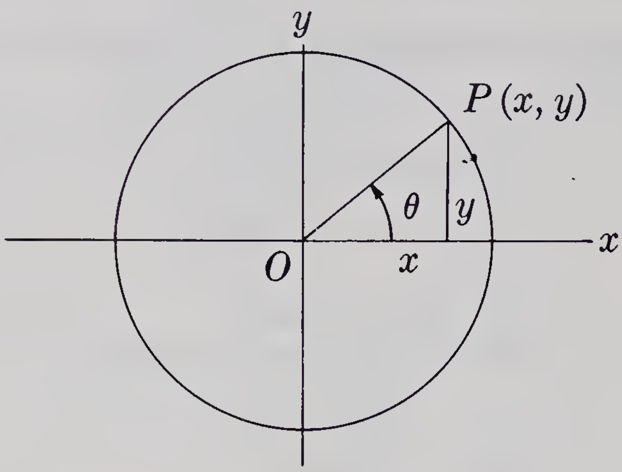


Figure 5.3

In Figure 5.3, if $P(x, y)$ is any point on a unit circle with centre at $(0, 0)$, then

$$\frac{x}{1} = \cos\theta\,,$$

or

$$x = \cos\theta\,,$$

and

$$y = \sin\theta\,.$$

By the theorem of Pythagoras,

$$x^2 + y^2 = 1\,,$$

or

$$\cos^2\theta + \sin^2\theta = 1\,.$$

Now

$$x^2 \geq 0, \quad \text{since} \quad x \in Re\,;$$

therefore,

$$y^2 \leq 1\,;$$

that is,

$$\sin^2\theta \leq 1\,,$$

and

$$-1 \leq \sin\theta \leq 1\,.$$

From our discussion, we conclude that the range for the sine function is the set of real numbers between -1 and $+1$, including the real numbers -1 and $+1$ themselves.

In summary, the sine function is defined by

$$f(\theta) = \sin\theta, \qquad \theta \in Re,$$

and the range is such that

$$|f(\theta)| \leq 1 .$$

Similarly, the cosine function is defined by

$$f(\theta) = \cos\theta, \qquad \theta \in Re,$$

and the range is such that

$$|f(\theta)| \leq 1 .$$

Returning to Figure 5.3, an examination of the ordinate of P as θ increases from 0 to 2π shows that y or $\sin\theta$ changes from 0 to 1 to 0 to -1 and back to 0. These values are incorporated in the following table of values.

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π	$\frac{9\pi}{4}$	$\frac{5\pi}{2}$	$\frac{11\pi}{4}$	3π	$\frac{13\pi}{4}$	$\frac{7\pi}{2}$	$\frac{15\pi}{4}$	4π
$\sin\theta$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{1}{\sqrt{2}}$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0

This table produces the points plotted in the right half of the graph in Figure 5.4. Similar values may be calculated for $-4\pi \leq \theta \leq 0$ and points plotted for the left half of the graph. We have used θ as the abscissa and $\sin\theta$ as the ordinate, and have joined the points by straight line segments. Despite this lack of accuracy, we can see the repeating or periodic nature of the sine function. The function takes all its values for $0 < \theta \leq 2\pi$ and this interval of 2π is called the period of the function.

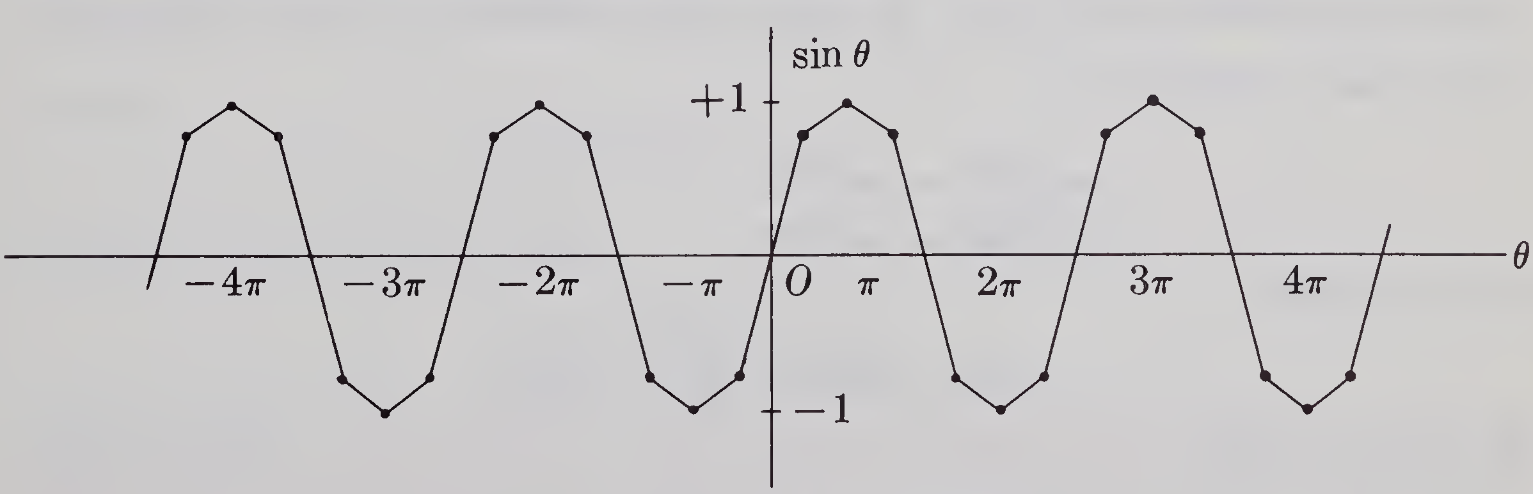


Figure 5.4

By plotting points obtained by calculating values for $\sin\theta$ at, say, 15° intervals for θ, we may obtain the more accurate smooth curve shown in Figure 5.5.

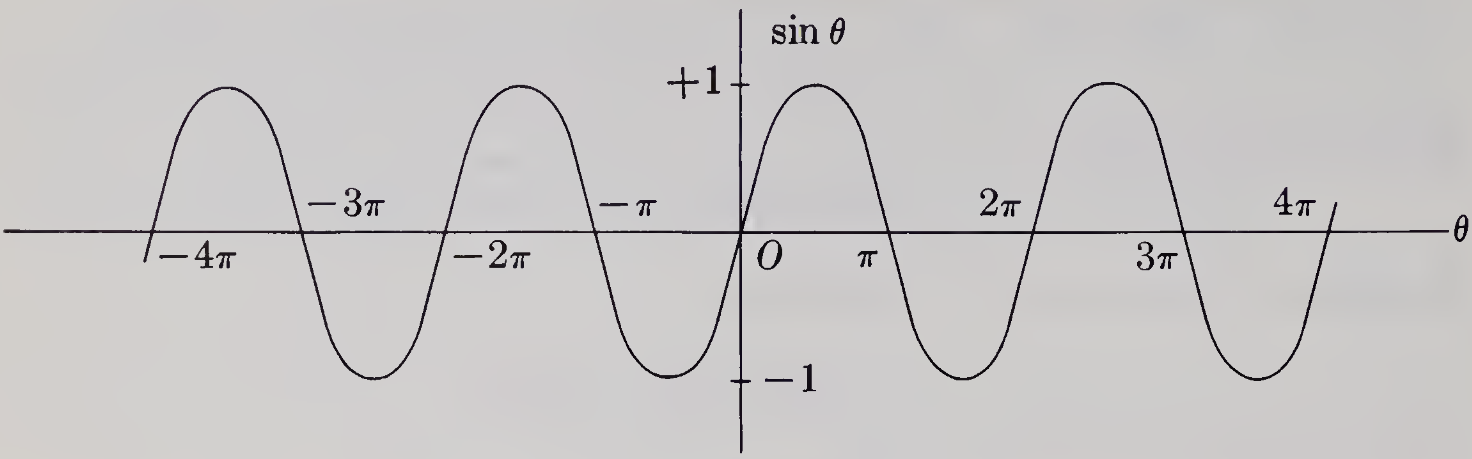


Figure 5.5

A similar graph for the cosine function is shown in Figure 5.6.

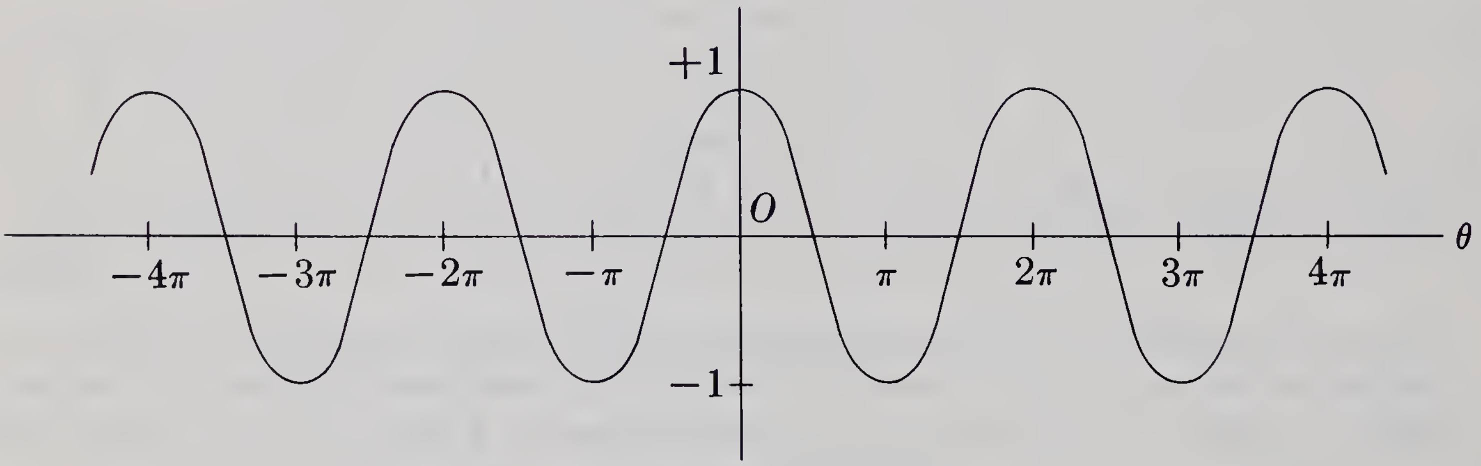


Figure 5.6

Note that the graph of the cosine function can be made to coincide with the graph of the sine function if we "move" the cosine curve a distance $\frac{\pi}{2}$ to the right parallel to the θ-axis. That is,

$$\sin 90^\circ = \cos 0^\circ,$$
$$\sin 180^\circ = \cos 90^\circ,$$

and, in general,

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right), \qquad \theta \in Re.$$

We may also write

$$\cos\theta = \sin\left(\theta + \frac{\pi}{2}\right), \qquad \theta \in Re.$$

From Figure 5.5, the maximum value of $\sin \theta$ is 1 and this maximum occurs when

$$\theta = \cdots, -\frac{7\pi}{2}, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, \cdots.$$

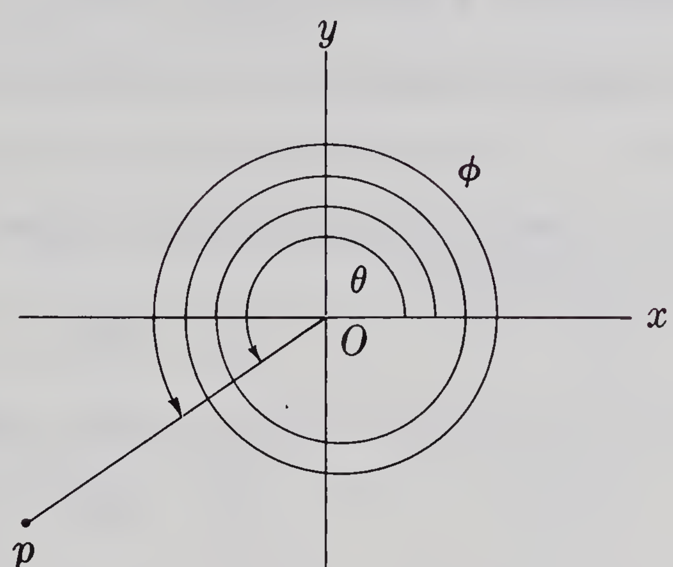
The minimum of -1 occurs when

$$\theta = \cdots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, \cdots.$$

The function takes zero values for

$$\theta = \cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots.$$

We have defined period from a consideration of the graph in Figure 5.5; let us examine it in a slightly different way.



Suppose that θ and ϕ are angles which differ by an integral multiple of 2π radians, as in the diagram above. Then these angles have the same final ray OP, hence the same value of the sine function. But

$$\phi = \theta + 2n\pi, \qquad n \in I;$$

therefore,

$$\sin(\theta + 2n\pi) = \sin \theta.$$

Similarly, it may be shown that

$$\cos(\theta + 2n\pi) = \cos \theta.$$

DEFINITION. The period of the sine or cosine function is 2π; that is, $\sin \theta = \sin(\theta + 2\pi)$ and $\cos \theta = \cos(\theta + 2\pi)$, for all $\theta \in Re$.
The period of a function f is the smallest real number, say T, for which the statement "$f(\theta) = f(\theta + T)$, for all $\theta \in Re$" is true.

EXERCISE 5.2

1. Construct a table of values for $\cos\theta$ from $0°$ to $360°$ at intervals of $15°$ for θ. Use the table to draw the graph of the cosine function for $-360° \leq \theta \leq 360°$.

2. From the graph in question (1), find the following.
   (a) The maximum value of $\cos\theta$.
   (b) The values of θ for which the maximum occurs.
   (c) The minimum value of $\cos\theta$.
   (d) The values of θ for which the minimum occurs.
   (e) The values of θ for which $\cos\theta = 0$.

3. Prove that, if θ and ϕ are two angles such that
$$\phi = \theta + 2n\pi, \qquad n \in I,$$
$$\cos\phi = \cos(\theta + 2n\pi) = \cos\phi, \qquad \theta \in Re.$$

4. What symmetry, if any, is possessed by (a) the sine function, (b) the cosine function?

5. Use the property of periodicity to give four values for θ so that $\sin\theta$ is equal to each of the following.
   (a) 0 (b) $-.5$ (c) $.7071$ (d) -1

6. Repeat question (5) so that $\cos\theta$ is equal to each of the following.
   (a) $-\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2}$ (c) $-.8660$ (d) $-.7660$

7. Consider the graphs of the sine and cosine functions in the interval $0 \leq \theta \leq 2\pi$. For what values of θ, if any, in this interval is each of the following equations a true statement?
   (a) $\sin\theta = \cos\theta$ (b) $\sin\theta = -\cos\theta$
   (c) $\sin\theta + \cos\theta = 1$ (d) $\sin\theta + \cos\theta = 2$
   (e) $\sin\theta + \cos\theta = -3$ (f) $\sin\theta = \cos\theta - 1$

8. The amplitude of a periodic function is one half the difference between the maximum and minimum ordinates. What is the amplitude of (a) the sine function $(\sin\theta)$, (b) the cosine function $(\cos\theta)$?

9. Repeat question (8) for the functions defined by (a) $y = 3\sin x$, (b) $y = k\cos x$.

10. Prove that, for any value of θ,
$$\sin(\theta + 2n\pi) = \cos\left(\theta - \frac{\pi}{2}\right).$$

5.3 Other Trigonometric Functions and Their Graphs

The tangent function whose value is given by $\tan \theta = \frac{y}{x}$, $x \neq 0$, has a domain that excludes those values for θ for which $x = 0$. These are

$$\theta = \cdots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \cdots.$$

The domain of the tangent function is

$$\left\{\theta \,\middle|\, \theta \in Re,\ \theta \neq \frac{\pi}{2} + n\pi,\ n \in I\right\}$$

and the range is Re, as there is no restriction on $\frac{y}{x}$ when x is not zero.

The following table of values has been used to construct the portion of the graph in Figure 5.7 for which $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. In this interval of π the function takes all real values. The period of the tangent function is π; that is,

$$\tan (\theta + n\pi) = \tan \theta, \qquad n \in I.$$

θ	$-\frac{\pi}{2}$	$-\frac{\pi}{3}$	$-\frac{\pi}{4}$	$-\frac{\pi}{6}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan \theta$	not defined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	not defined

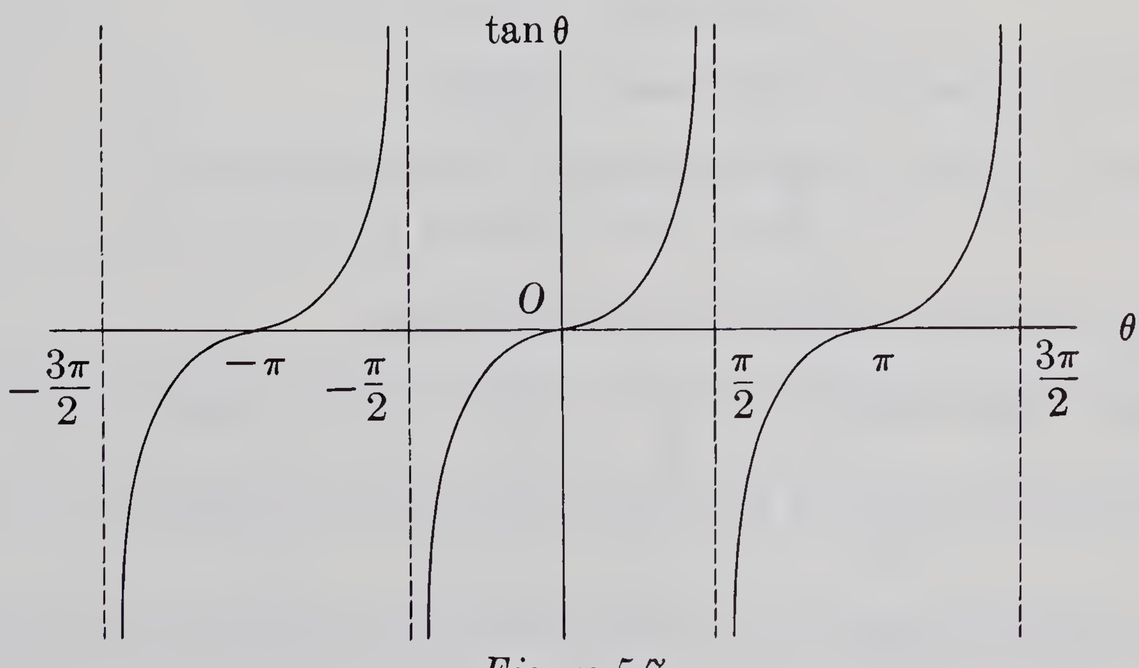


Figure 5.7

There are no maximum or minimum values. The zeros of the function occur at

$$\theta = \cdots, -2\pi, \ -\pi, 0, \pi, 2\pi, \cdots.$$

The cotangent, secant, and cosecant functions are called reciprocal functions with respect to the tangent, cosine, and sine functions respectively. (Why?) The graphs of these functions are required in Exercise 5.3. In this exercise, it may be helpful to remember the identities

$\cot x = \dfrac{1}{\tan x}$, $\sec x = \dfrac{1}{\cos x}$, and $\csc x = \dfrac{1}{\sin x}$, in constructing tables of values.

EXERCISE 5.3

1. Using a table of values similar to that in Section 5.3 for $\tan\theta$, construct the graph of the cotangent function for
$$-\frac{3\pi}{2} < \theta < \frac{3\pi}{2}.$$

2. (a) Name the maximum and minimum values, if any, of the cotangent function.
   (b) State the domain and range of the function.
   (c) Name the values of θ at which the zeros of the function occur.
   (d) What is the period of the cotangent function?
   (e) Use the property of periodicity to name four values of θ for which $\cot\theta = 1$.

3. Calculate a suitable table of values and draw the graph of the secant function defined by
$$f(\theta) = \sec\theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi\,.$$

4. Repeat question (2) for the secant function.

5. Repeat question (3) for the cosecant function defined by
$$f(\theta) = \csc\theta \quad \text{for} \quad 0 < \theta < 2\pi\,.$$

6. Repeat question (2) for the cosecant function.

7. How does the graph of $y = 3\tan x$ differ from the graph of $y = \tan x$?

8. How does the graph of $y = \dfrac{1}{2}\sec x$ differ from the graph of $y = \sec x$?

9. How does the graph of $y = \sec\dfrac{x}{2}$ differ from the graph of $y = \sec x$?

10. Draw a graph of the function $y = \sin^2 x$ for $0 \leq x \leq 2\pi$.

11. Draw a graph of the function $y = \cos^2 x$ for $-2\pi \leq x \leq 2\pi$.

5.4 The Amplitude of $a \sin \theta$

Consider the function defined by

$$y = a \sin \theta ,$$

where a is some given positive real number. We have, in fact, already considered one case and drawn its graph, the case where $a = 1$. If we look back at this graph, we see that $|y| \leq 1$ for all values of θ, and that $|y| = 1$ only at the points where y has a maximum or minimum value.

To see what happens to the graph of $\{(\theta, a \sin \theta) \mid \theta \in Re\}$ when $a \neq 1$, consider the case of $a = 2.5$. We calculate the following table of values:

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π
$\sin \theta$	0	.5000	.7071	.8660	1	.8660	.7071	.5000	0
$2.5 \sin \theta$	0	1.25	1.7677	2.1650	2.5	2.1650	1.7677	1.25	0

θ	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sin \theta$	0	$-.5000$	$-.7071$	$-.8660$	-1	$-.8660$	$-.7071$	$-.5000$	0
$2.5 \sin \theta$	0	-1.25	-1.7677	-2.1650	-2.5	-2.1650	-1.7677	-1.25	0

Obviously, each value in the last row is 2.5 times the corresponding value in the row above. The graph is shown in Figure 5.8.

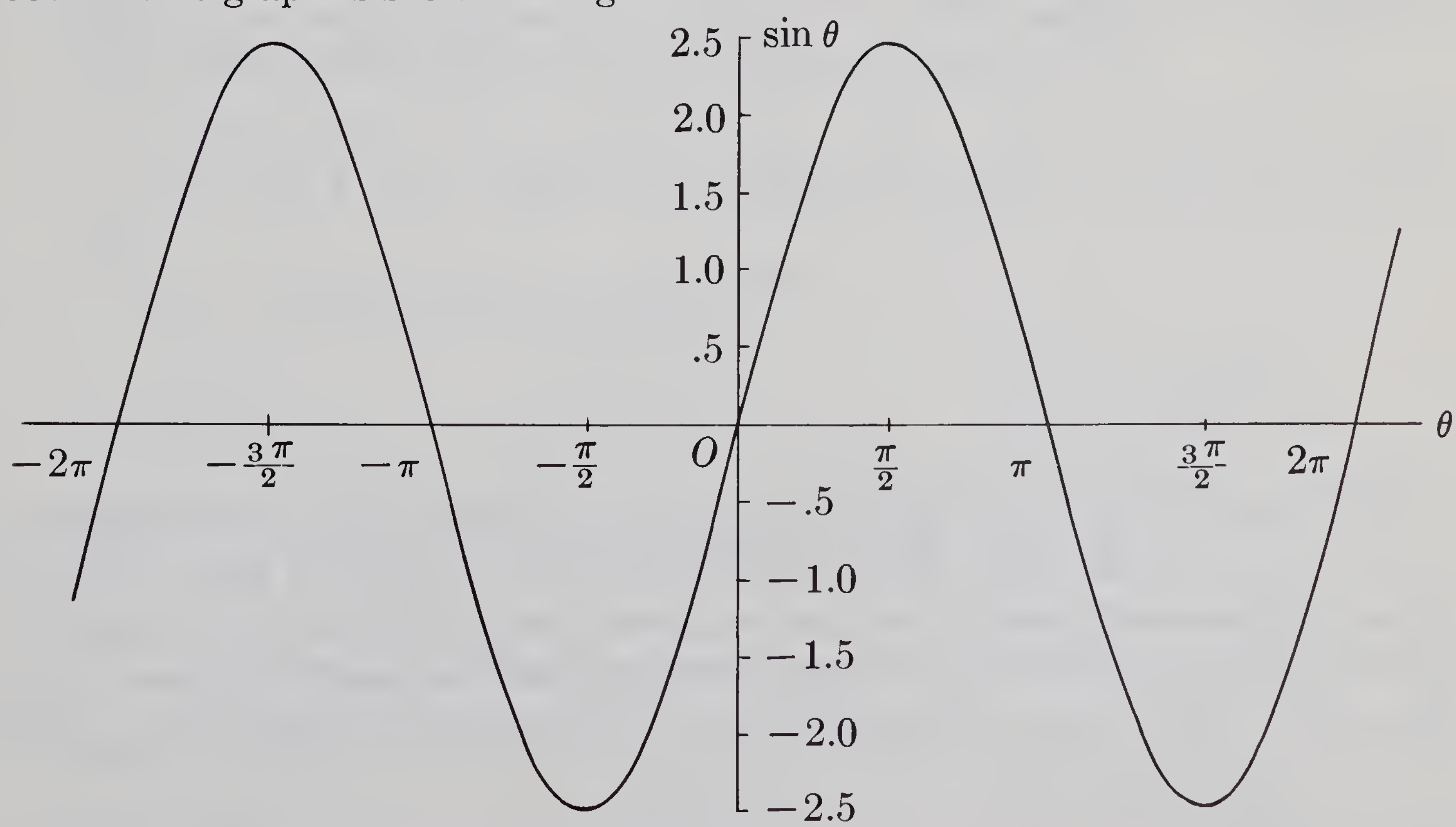


Figure 5.8

We see, either from the graph or from the table of values, that, for all values of θ, $|y| \leq 2.5$, and $|y| = 2.5$ only at the maximum and minimum points.

If we draw the graphs for other values of $a > 1$, we observe similar results: $|y| \leq a$ for all values of the argument θ and $|y| = a$ only at the maxima and minima.

In these cases, we notice also that the graph of $a \sin \theta$ is very much like that of $\sin \theta$. It has zeros for the same values of the argument θ $(\cdots, -2\pi, -\pi, 0, +\pi, +2\pi, \cdots)$ and its maxima and minima occur at the same values of the argument θ. The distinction between the two graphs is that the value of y for any given value of θ is greater; in fact, it is a times as great (see Figure 5.8). In particular, the maxima and minima are a times as far from the θ-axis. We might say that the graph of $a \sin \theta$ is the graph of $\sin \theta$ with the ordinate magnified, or amplified, to use a term from sound rather than sight. Thus a is an amplification factor; it is used very frequently in science and is called *the amplitude.*

So far, we have only discussed the cases when $a = 1$ and when $a > 1$. The case when $a < 1$ is very similar. At the maxima and minima, $|y| = a$ as before, but now all the ordinates are *reduced* in the ratio $a:1$. Figure 5.9 shows the graph of the equation $y = a \sin \theta$ for the case $a = .4$.

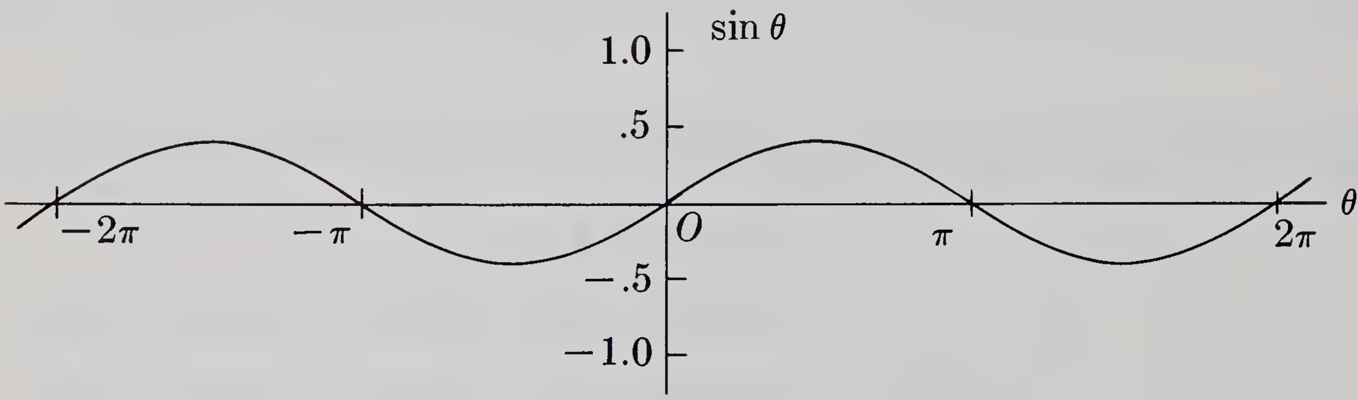


Figure 5.9

When $a < 1$ and $y = a \sin \theta$, we might be tempted to call a the reduction factor or some other name which would indicate a decrease in size. However mathematicians and scientists use the same name, *the amplitude,* for *all* values of a and distinguish between the two cases by noting whether a is greater than, or less than, one. Notice that in the case of $a = 1$, the function is defined by $y = \sin \theta$ and the amplitude is equal to 1.

DEFINITION. If $y = a \sin \theta$, then $|a|$ is called the amplitude of the function.

Figure 5.10 illustrates the cases $a = 2.5$, 1.0, and $.4$ on the same diagram for comparison purposes. Now that we know the shape of the graph of the sine function we can sketch such curves and we plot only the positions of the zeros, the maxima, and the minima.

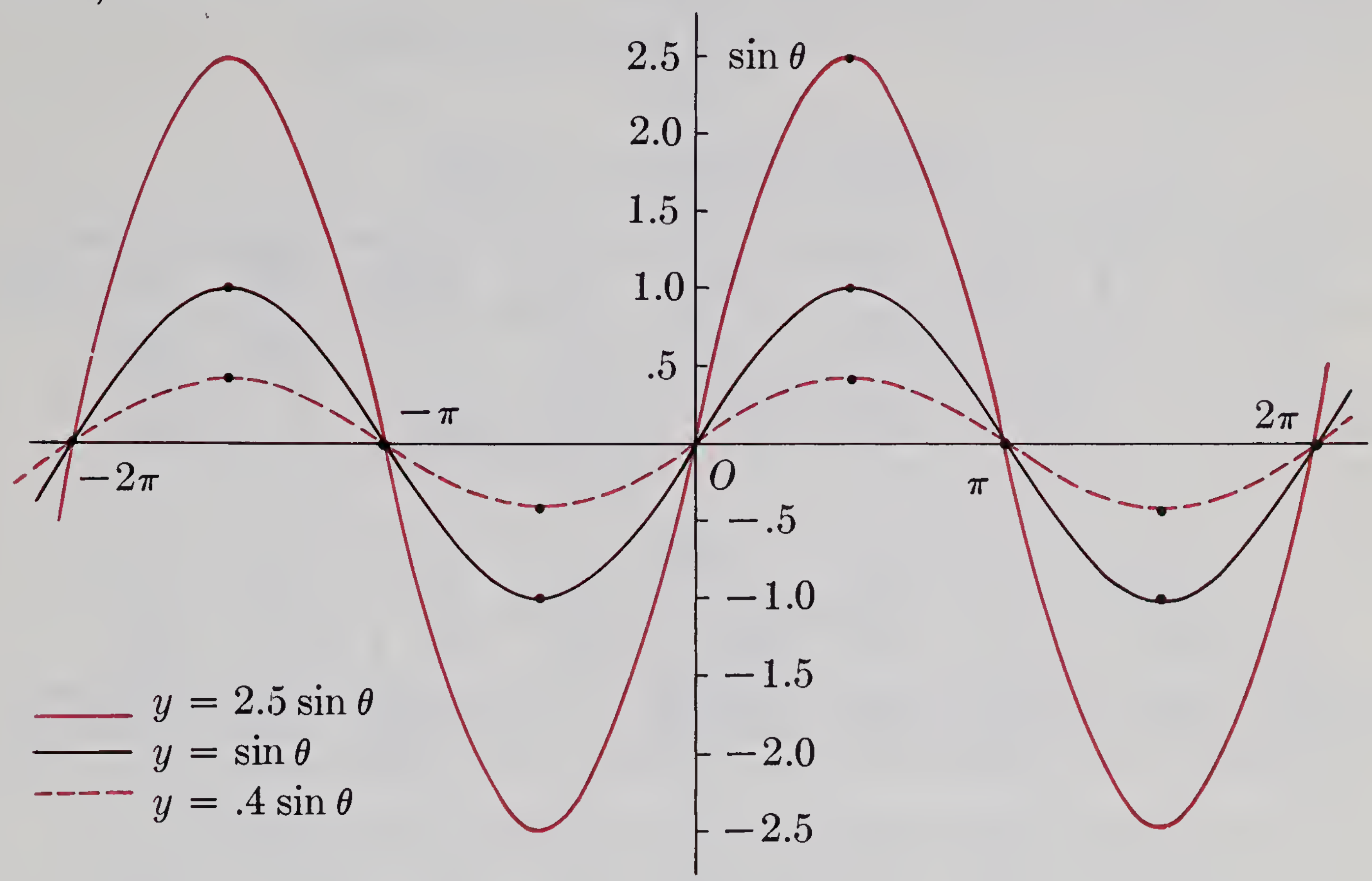


Figure 5.10

EXERCISE 5.4

1. Calculate a table of values for $(\theta, 2\sin\theta)$ at 30° intervals for θ between 0° and 180°, and draw the graph between −360° and 360°.

2. Sketch the graphs of $y = 0.5\sin\theta$, $y = \sin\theta$, and $y = 3\sin\theta$, between $\theta = -2\pi$ and $\theta = 2\pi$, relative to the same set of axes. Compare the amplitudes.

3. State the amplitude of the following:

(a) $3\sin\theta$	(b) $\frac{1}{5}\sin\theta$	(c) $= 4.3\sin\theta$
(d) $.002\sin\theta$	(e) $2\pi\sin\theta$	(f) $= 3 \cdot 10^6 \sin\theta$

4. Calculate a table of values for $(\theta, 1.5\cos\theta)$ at 30° intervals from 0° to 90°. Draw the graph of $y = 1.5\cos\theta$ between $\theta = 0°$ and 720°.

5. Sketch the graphs of $y = a\cos\theta$ with $a = .8$, 1, and 2.0 between $\theta = -2\pi$ and $\theta = 2\pi$. Compare the amplitudes.

6. State the amplitude of the following functions.

(a) $7\cos\theta$	(b) $.03\cos\theta$	(c) $5 \cdot 10^3 \cos\theta$
(d) $6 \cdot 10^{-2}\cos\theta$	(e) $\sqrt{2}\cos\theta$	(f) $\frac{2}{5}\cos\theta$

5.5 The Periodicity of $\sin k\theta$ and $\cos k\theta$

Another interesting case of great importance in science occurs when the argument is $k\theta$, where k is a positive real number.

The function defined by $y = \sin 3\theta$ is used as an example.

Table of Values for $y = \sin 3\theta$

θ	0°	5°	10°	15°	20°	25°	30°	35°
3θ	0°	15°	30°	45°	60°	75°	90°	105°
$\sin 3\theta$	0	.2588	.5000	.7071	.8660	.9659	1	.9659

θ	40°	45°	50°	55°	60°	65°	70°	75°
3θ	120°	135°	150°	165°	180°	195°	210°	225°
$\sin 3\theta$	.8660	.7071	.5000	.2588	0	−.2588	−.5000	−.7071

θ	80°	85°	90°	95°	100°	105°	110°	115°
3θ	240°	255°	270°	285°	300°	315°	330°	345°
$\sin 3\theta$	−.8660	−.9659	−1	−.9659	−.8660	−.7071	−.5000	−.2588

θ	120°	125°	130°	135°	140°	145°	150°	155°
3θ	360°	375°	390°	405°	420°	435°	450°	465°
$\sin 3\theta$	0	.2588	.5000	.7071	.8660	.9659	1	.9659

θ	160°	165°	170°	175°	180°
3θ	480°	495°	510°	525°	540°
$\sin 3\theta$	.8660	.7071	.5000	.2588	0

The graph for $k = 1$, that is $y = \sin \theta$, is sketched on the same scale for comparison. The amplitude is not changed; it is *one* in both cases. However the maxima and minima are closer together when $k = 3$ than when $k = 1$, and the same applies to the zeros. In fact the whole graph has been *squeezed* in the direction parallel to the θ-axis. Therefore, the periods of the two functions must be different.

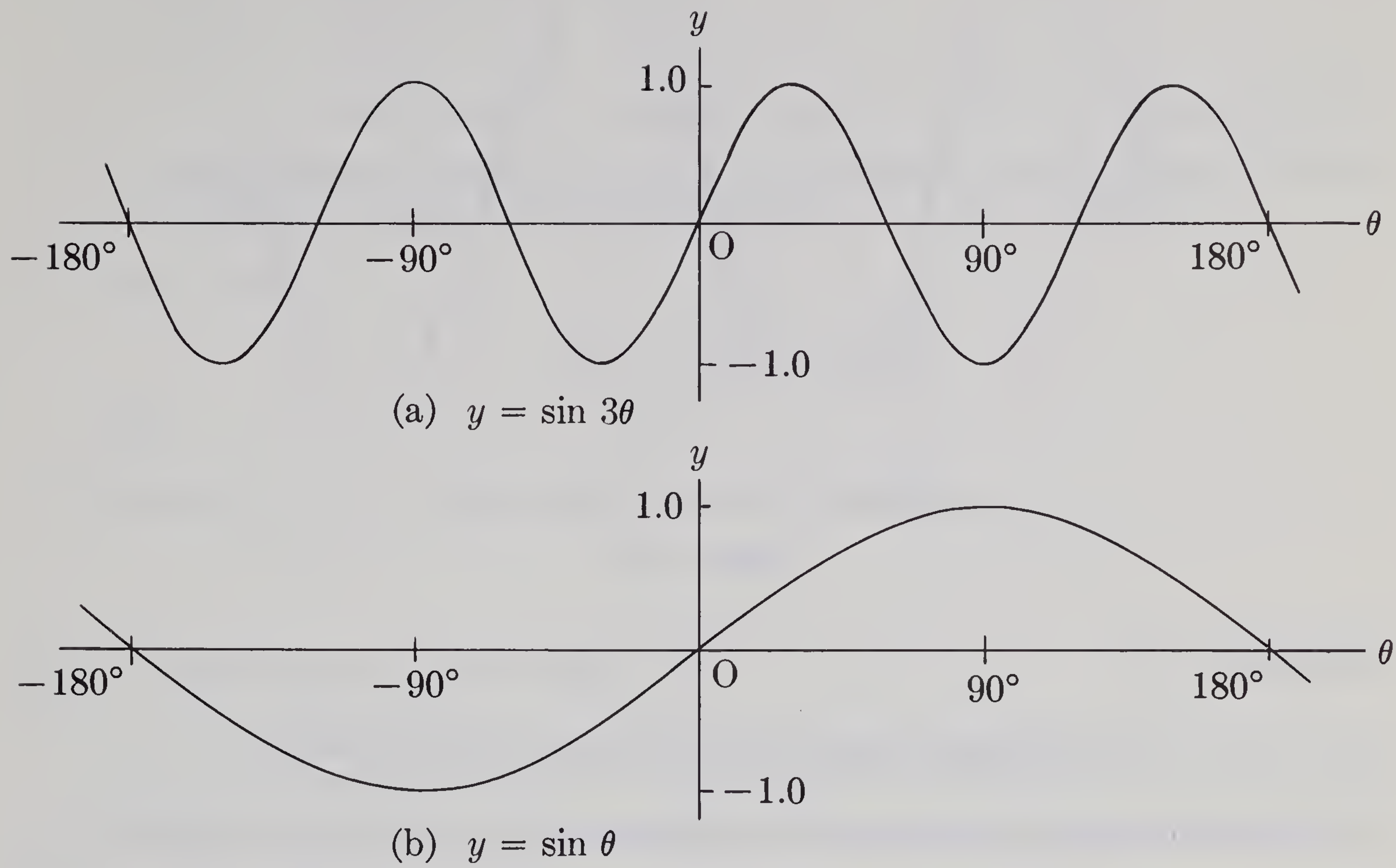


Figure 5.11

DEFINITION. The *period, in θ,* of the sine function, $\sin k\theta$, is the difference in the value of θ between two adjacent zeros of $y = \sin k\theta$ at which y changes sign from negative to positive as θ increases.

If we examine Figure 5.11 (a) and (b), we see that for $y = \sin 3\theta$, the period is $\frac{2\pi}{3}$, and for $y = \sin \theta$, the period is 2π.

Note that if θ alone is the argument of the sine function, we usually shorten the phrase 'the period in θ' to merely 'the period'.

Thus, the period in θ is only one third as great for $y = \sin 3\theta$ as for $y = \sin \theta$. Alternatively, we could say that $y = \sin 3\theta$ repeats 3 times as rapidly when θ increases as does $y = \sin \theta$.

When you sketch the graphs for other values of k in the exercises, you will see in every case the period in θ is $\frac{2\pi}{k}$.

Alternative Definition: The period in θ of the sine function $\sin k\theta$ is $\frac{2\pi}{k}$.

Sketch graphs for the cases $k = 1, 3$, and $.5$ are superimposed for comparison in Figure 5.12.

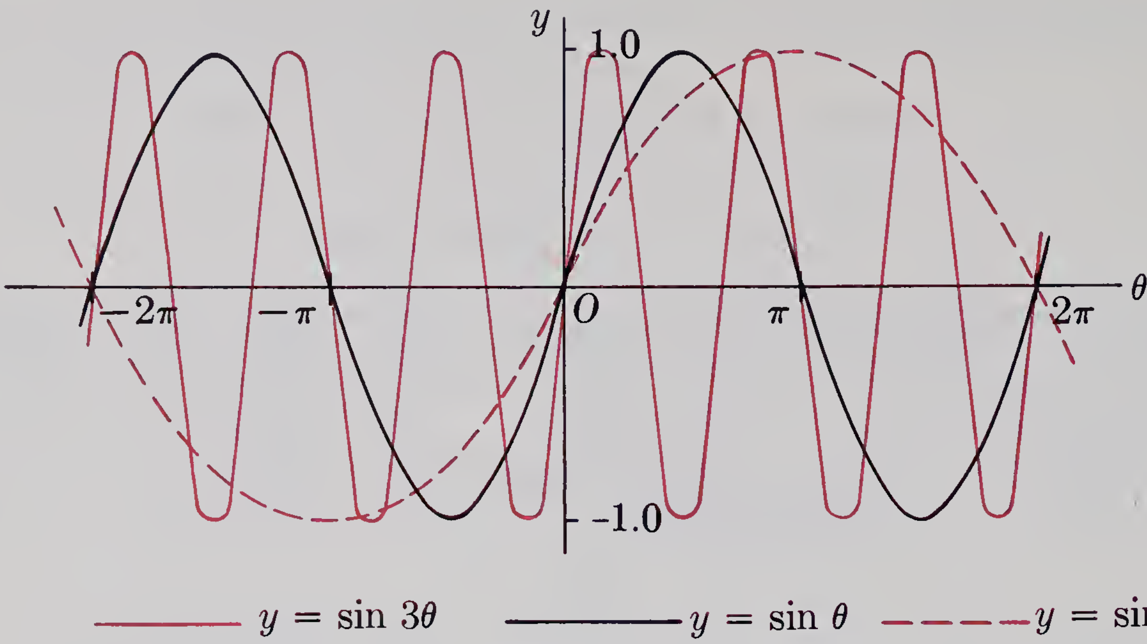


Figure 5.12

EXERCISE 5.5

(*In all graphs show at least two periods of the function.*)

1. Sketch the graphs of the functions determined by the following equations.

   (a) $y = \sin 2\theta$ (b) $y = \sin 4\theta$ (c) $y = \sin \dfrac{\theta}{2}$ (d) $y = \sin \dfrac{\theta}{4}$.

   State the periods in θ of these functions.

2. Sketch the graphs of the functions determined by the following equations.

   (a) $y = \sin 3\theta$ (b) $y = \sin 4\theta$

   relative to the same set of axes. When does the combined pattern formed by the two functions begin to repeat for the first time for positive θ?

3. Sketch the graphs of the functions determined by the following equations.

   (a) $y = \cos \theta$ (b) $y = \cos 2.5\theta$ (c) $y = \cos 5\theta$ (d) $y = \cos .4\theta$.

   State the periods in θ of these functions.

4. Repeat question (2) for the curves given by $y = \cos 2\theta$ and $y = \cos 5\theta$.

5. Examine the graphs of the pairs of functions defined as follows:

   (a) $y = \sin 2\theta$, $y = \sin 4\theta$ (b) $y = \sin \dfrac{\theta}{2}$, $y = \sin 2\theta$

   (c) $y = \cos \dfrac{\theta}{3}$, $y = \cos \dfrac{\theta}{2}$ (d) $y = \cos 2\theta$, $y = \cos \dfrac{\theta}{3}$

   What do you notice about the period of the pattern of the pairs and the individual periods?

6. (a) Show that the difference in the values of θ between two adjacent zeros of $y = \sin k\theta$ at which y changes sign from positive to negative is equal to the period in θ.
   (b) Show that the period in θ of
   $$y = \sin k\theta$$
   is equal to the difference between the values of θ at two adjacent maxima.
   (c) Show that the period in θ of $y = \sin k\theta$ is equal to the difference between the values of θ at two adjacent minima.
7. Prove that
$$\sin k\theta = \sin k\left(\theta + n\frac{2\pi}{k}\right)$$
and
$$\cos k\theta = \cos k\left(\theta + n\frac{2\pi}{k}\right).$$

5.6 The Phase of sin $(\theta + d)$

If we draw the graph of $y = 2 \sin\left(\theta + \frac{\pi}{3}\right)$, we can compare it with the graph of $y = 2 \sin \theta$ which we studied in Section 5.4. Figure 5.13 shows the two graphs drawn to the same axes for comparison.

Table of values of $y = \sin\left(\theta + \frac{\pi}{3}\right)$

θ	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
$\theta + \frac{\pi}{3}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$
$\sin\left(\theta + \frac{\pi}{3}\right)$	.8660	1.0	.8660	.5000	0	$-.5000$	$-.8660$

θ	$\frac{7\pi}{6}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π
$\theta + \frac{\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{11\pi}{6}$	2π	$\frac{13\pi}{6}$	$\frac{7\pi}{3}$
$\sin\left(\theta + \frac{\pi}{3}\right)$	-1.0	$-.8660$	$-.5000$	0	.5000	.8660

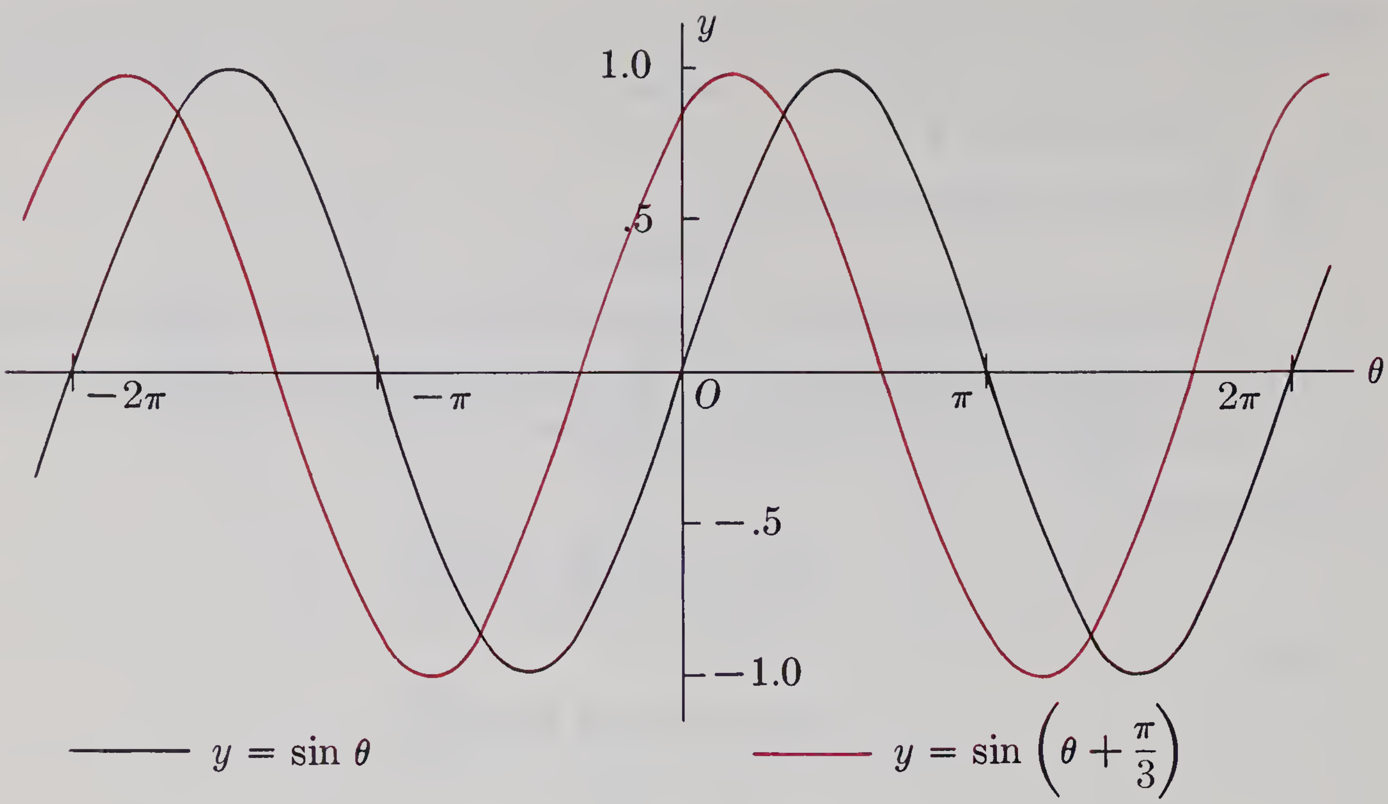


Figure 5.13

We notice immediately that the relationship between these two curves resembles that between $y = \sin \theta$ and $y = \cos \theta$, which we discussed in Section 5.2. The two curves have the same shape and size but are moved or shifted relative to each other in the direction of the θ-axis.

DEFINITIONS. If $y = \sin (\theta + d)$, then the constant angle d is called the phase angle of the function with respect to $y = \sin \theta$.

The relative shift of the curve $y = \sin (\theta + d)$ with respect to $y = \sin \theta$ is called a phase shift.

Generally the range of d is restricted so that $-\pi \leq d \leq \pi$.

(If $y = a \cos (\theta + d)$, then d is the phase angle of the function with respect to $y = a \cos \theta$).

If we examine the two graphs shown we see that

the zeros of $y = \sin \theta$ are $\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\}$, and

the zeros of $y = \sin \left(\theta + \frac{\pi}{3}\right)$ are $\left\{\ldots, -\frac{7\pi}{3}, -\frac{4\pi}{3}, -\frac{\pi}{3}, \frac{2\pi}{3}, \frac{5\pi}{3} \ldots\right\}$. Thus the phase angle, $\frac{\pi}{3}$, determines the displacement of the zeros of $y = \sin \left(\theta + \frac{\pi}{3}\right)$ with respect to those of $y = a \sin \theta$. This displacement is most easily calculated by finding the value of θ for which the argument is zero; this zero is at $\theta + \frac{\pi}{3} = 0$, that is, $\theta = -\frac{\pi}{3}$, and the phase shift is $\frac{\pi}{3}$ to the left.

In the general case of $y = \sin (\theta + d)$, the phase shift is given by $\theta = -d$, which is the condition for the argument of the sine function to be zero.

If d is positive $\left(\text{in our example } d = \frac{\pi}{3}\right)$, the function $y = \sin\,(\theta + d)$ is said to *lead* the function $y = \sin\,\theta$; the corresponding zeros of $y = \sin\,(\theta + d)$ occur *before or to the left* of the zeros of $y = \sin\,\theta$.

If d is negative, then $y = \sin\,(\theta + d)$ is said to *lag* the function $y = \sin\,\theta$. Note that in this terminology

$$y = \cos\theta = \sin\left(\theta + \frac{\pi}{2}\right) \text{ leads } y = \sin\theta \text{ by } \frac{\pi}{2},$$

and

$$y = \sin\theta = \cos\left(\theta - \frac{\pi}{2}\right) \text{ lags } y = \cos\theta \text{ by } \frac{\pi}{2}.$$

Note also that a lead by π has the same effect as a lag by π; $\sin\,\theta$ and $\sin\,(\theta \pm \pi)$ are said to be in antiphase.
The phase angle is a property of the trigonometric functions that is independent of the amplitude.

EXERCISE 5.6

1. Draw the graph of *one* of the functions defined as follows and compare it with the graph of

$$y = \sin\theta.$$

Sketch the other graphs.

(a) $y = \sin\left(\theta + \frac{\pi}{6}\right)$ (b) $y = \sin\left(\theta - \frac{3\pi}{4}\right)$

(c) $y = \sin\,(\theta + 25°)$ (d) $y = \sin\,(\theta - 120°)$

State the phase angle with respect to $y = \sin\,\theta$ in each case.

2. Sketch the graphs of the functions defined as follows and compare them with the graph of

$$y = a\cos\theta,$$

where a is the same amplitude in each case as that of the given function:

(a) $y = \cos\left(\theta + \frac{\pi}{4}\right)$ (b) $y = \cos\left(\theta - \frac{2\pi}{3}\right)$

(c) $y = \cos\,(\theta - 75°)$ (d) $y = \cos\,(\theta + 150°)$

State the amplitude and phase angle with respect to $y = \cos\,\theta$ in each case.

3. Give the phase shift relative to $\sin\,\theta$ for each of the following:

(a) $\sin\left(\theta + \frac{\pi}{3}\right)$ (b) $\sin\left(\theta - \frac{3\pi}{4}\right)$ (c) $\sin\,(\theta - 144°)$

(d) $\sin\left(\theta - \frac{\pi}{\sqrt{2}}\right)$ (e) $\sin\,(\theta + \sqrt{2})$ (f) $\sin\left(\theta + \frac{\pi}{4}\right)$

4. At what values of θ in the interval $-4\pi < \theta < 4\pi$ or $-720° < \theta < 720°$, are the following equal to zero?

(a) $\sin\left(\theta + \frac{2\pi}{3}\right)$ (b) $\cos\left(\theta - \frac{\pi}{4}\right)$ (c) $\cos\left(\theta + \frac{\pi}{6}\right)$

(d) $\sin(\theta - 63°)$ (e) $\cos(\theta + 40°)$ (f) $\sin(\theta - 60°)$

5. For what values of θ in the domain $0 < \theta < 6\pi$ or $0 < \theta < 1080°$ do the following attain their maximum values?

(a) $\sin(\theta + 45°)$ (b) $\cos\left(\theta - \frac{\pi}{3}\right)$ (c) $\cos\left(\theta + \frac{3\pi}{4}\right)$

(d) $\sin\left(\theta + \frac{5\pi}{6}\right)$ (e) $\cos(\theta - 125°)$ (f) $\sin\left(\theta - \frac{2\pi}{3}\right)$

For what values of θ do the minima occur?

5.7 The Functions $a \sin(k\theta + d)$ and $a \cos(k\theta + d)$

We can now study the more general function defined by

$$y = a \sin(k\theta + d),$$

where a is a positive real number,
k is a positive real number, and
d is the phase angle, a real number such that $-\pi \leq d \leq \pi$.

The graph of a particular case, $y = 2.5 \sin(3\theta + \frac{1}{4}\pi)$, is given in Figure 5.14 with the graph of $y = 2.5 \sin 3\theta$ for comparison.

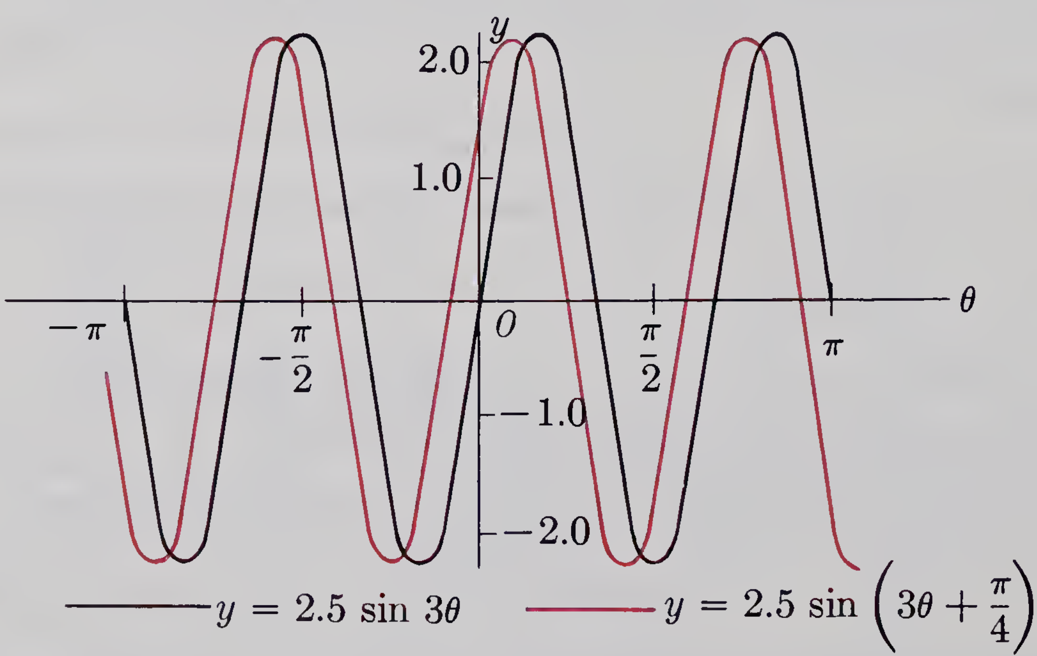


Figure 5.14

Both functions have the same amplitude and the same period in θ, but they are different in phase.

The phase angle of $y = 2.5 \sin\left(3\theta + \frac{\pi}{4}\right)$ is $\frac{\pi}{4}$. Note, however, that this is not now the shift of the graph to the left from the graph of $y = 2.5 \sin 3\theta$; the left shift is given by $3\theta + \frac{\pi}{4} = 0$, that is, $\theta = -\frac{\pi}{12}$.

The graphs of three other similar functions determined by (a) $y = 1.5 \sin\left(3\theta + \frac{\pi}{4}\right)$, (b) $y = 2.5 \sin\left(2\theta + \frac{\pi}{4}\right)$, and (c) $y = 2.5 \sin\left(3\theta + \frac{2\pi}{3}\right)$ are given in Figure 5.15 for comparison.

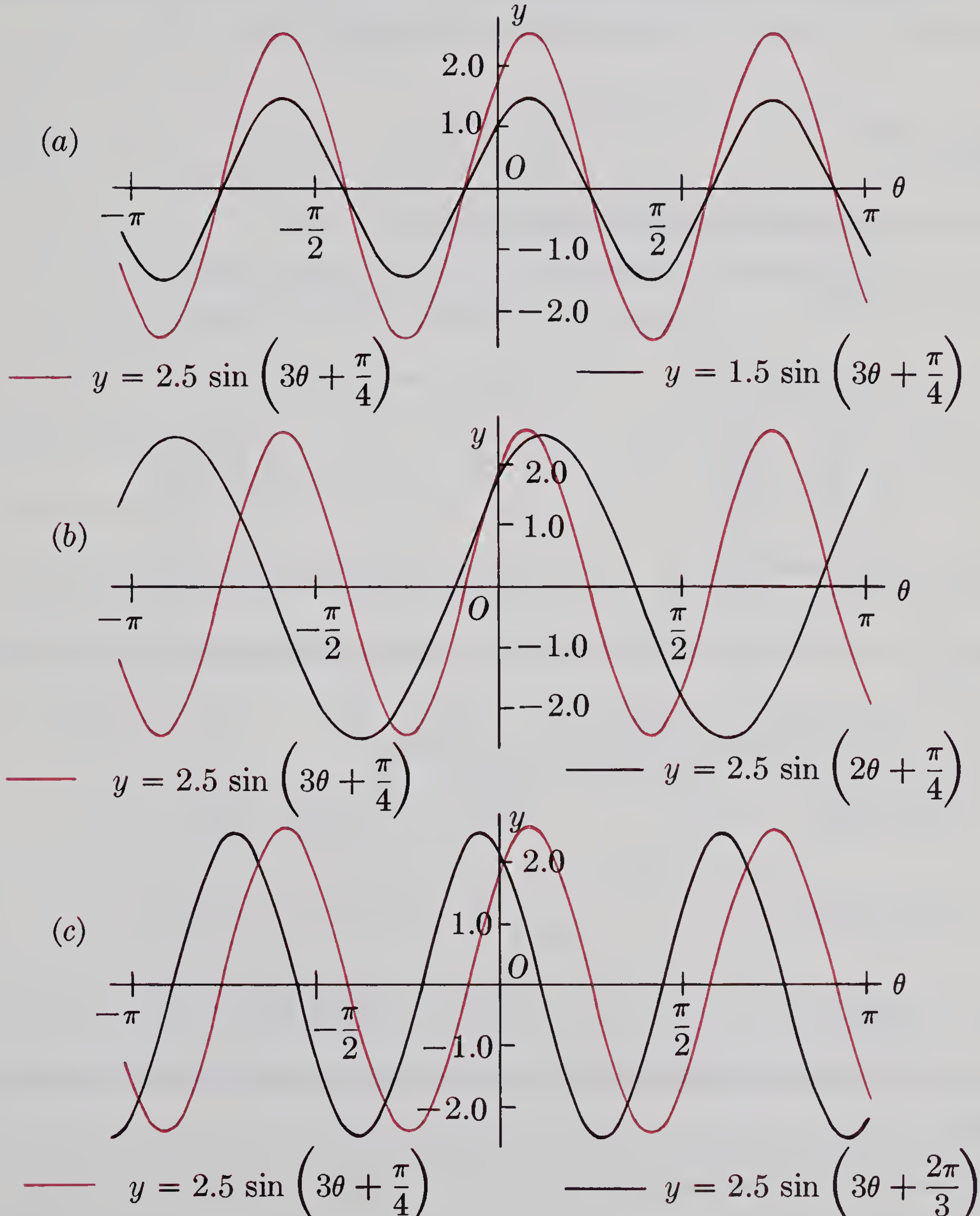


Figure 5.15

From these graphs it follows that

(i) a change in the amplitude leaves the period in θ and phase angle unchanged,

(ii) a change in the period in θ leaves the amplitude and phase angle unchanged, and

(iii) a change in the phase angle leaves the amplitude and period in θ unchanged.

The amplitude, period in θ, and phase angle are thus independent properties of the functions; however, the phase shift in θ is dependent on both the phase angle and the period in θ, but it is independent of the amplitude.

EXERCISE 5.7

1. Sketch the graphs of the functions defined by

   (a) $y = .5 \sin \theta$ (b) $y = \sin\left(\theta + \frac{\pi}{4}\right)$

   (c) $y = .5 \sin\left(\theta + \frac{\pi}{4}\right)$ (d) $y = .5 \sin 3\theta$

   (e) $y = \sin 3\left(\theta + \frac{\pi}{4}\right)$ (f) $y = .5 \sin 3\left(\theta + \frac{\pi}{4}\right)$

   (g) $y = \sin\left(3\theta + \frac{\pi}{4}\right)$ (h) $y = .5 \sin\left(3\theta + \frac{\pi}{4}\right)$

   State the amplitude, periodicity, and the phase angle and shift of each relative to $\sin \theta$.

2. Sketch the graphs of the functions defined by

   (a) $y = \cos \theta$ (b) $y = \cos\left(\theta - \frac{2\pi}{3}\right)$

   (c) $y = 2 \cos \theta$ (d) $y = 2 \cos\left(\theta - \frac{2\pi}{3}\right)$

   (e) $y = \cos 4\theta$ (f) $y = 2 \cos 4\left(\theta - \frac{2\pi}{3}\right)$.

   State the amplitude, periodicity, and phase angle and shift of each relative to $\cos \theta$.

3. What is the phase angle and shift of $y = -2 \sin \theta$ relative to $y = \sin \theta$?

4. What is the phase angle and shift of $y = \sin (3\theta + \pi)$ relative to $y = \sin \theta$?

Chapter Summary

Definitions of the trigonometric functions · Functions of any angle · Domain, range, and graphs of the trigonometric functions · Amplitude and period of a function

The sine and cosine functions of θ exist in the domain of the set of real numbers, Re, and are subject to the inequalities $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$, respectively.

The domain of the functions defined by $y = \csc\theta$ and $y = \tan\theta$ is $\{\theta \in Re \mid \theta \neq n\pi,\ n \in I\}$.

The domain of the functions defined by $y = \sec\theta$ and $y = \cot\theta$ is $\left(\theta \in Re \mid \theta \neq \frac{(2n-1)\pi}{2},\ n \in I\right)$.

The range of $y = \sec\theta$ and $y = \csc\theta$ is $\{y \in Re \mid y \leq -1 \text{ or } y \geq 1\}$.

The range of $y = \tan\theta$ and $y = \cot\theta$ is Re.

$\sin(-\theta) = -\sin\theta$, $\cos(-\theta) = \cos\theta$, $\tan(-\theta) = -\tan\theta$.

The functions $\sin\theta$, $\cos\theta$, $\csc\theta$ and $\sec\theta$ are periodic with period 2π.

The functions $\tan\theta$ and $\cot\theta$ are periodic with period π.

The zeros of $\sin\theta$ are $\{\ldots, -2\pi, -\pi, 0, \pi, 2\pi, \ldots\} = \{n\pi,\ n \in \mathrm{I}\}$.

The zeros of $\cos\theta$ are $\left\{\ldots, -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2} \ldots\right\} = \left\{\left(\frac{n}{2} + 1\right)\pi,\ n \in \mathrm{I}\right\}$.

The zeros of $\tan\theta$ are $\{\cdots, -2\pi, -\pi, 0, \pi, 2\pi, \cdots\} = \{n\pi \mid n \in I\}$.

The zeros of $\cot\theta$ are $\left\{\cdots, -\frac{3}{2}\pi, -\frac{1}{2}\pi, \frac{1}{2}\pi, \frac{3}{2}\pi, \cdots\right\} = \left\{\frac{(2n-1)\pi}{2} \middle| n \in I\right\}$.

There are no zeros of $\csc\theta$ and $\sec\theta$.

$\sin\theta = 1$ when $\theta = \left\{\ldots, -\frac{3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \ldots\right\} = \left\{\left(2n + \frac{1}{2}\right)\pi,\ n \in \mathrm{I}\right\}$.

$\sin\theta = -1$ when $\theta = \left\{\ldots, -\frac{5\pi}{2}, -\frac{\pi}{2}, \frac{3\pi}{2}, \ldots\right\} = \left\{\left(2n - \frac{1}{2}\right)\pi,\ n \in \mathrm{I}\right\}$.

$\cos\theta = 1$ when $\theta = \{\ldots, -2\pi, 0, 2\pi, \ldots\} = \{2n\pi,\ n \in \mathrm{I}\}$.

$\cos\theta = -1$ when $\theta = \{\ldots, -3\pi, -\pi, \pi, 3\pi, \ldots\} = \{(2n+1)\pi,\ n \in \mathrm{I}\}$.

For the function determined by $y = a\sin(\theta + d)$,
a is the amplitude, 2π is the period in θ, and d is the phase angle.

For the functions determined by $y = a\sin(k\theta + d)$ and $y = a\cos(k\theta + d)$
a is the amplitude, $\frac{2\pi}{k}$ is the period in θ, d is the phase angle, and the phase shift in θ is $-\frac{d}{k}$.

REVIEW EXERCISE 5

Use tables to find the following values.

1. $\cos 39°$
2. $\sin 42°$
3. $\tan 14°$
4. $\cos 219°$
5. $\sin (-42°)$
6. $\tan 346°$
7. $\tan \frac{\pi}{10}$
8. $\cos \left(-\frac{\pi}{12}\right)$
9. $\sin \frac{\pi}{5}$
10. $\tan \frac{21\pi}{10}$
11. $\cos \frac{13\pi}{12}$
12. $\sin \frac{9\pi}{5}$

Without using tables, find the following values.

13. $\sin 60°$
14. $\sin 300°$
15. $\tan 315°$
16. $\tan 1215°$
17. $\cos (-45°)$
18. $\cos 22\frac{1}{2}°$
19. $\tan 67\frac{1}{2}°$
20. $\sec (-1500°)$
21. $\cos 10°$
22. Sketch the graphs of
    (a) $\tan 2x$ (b) $\cot \frac{1}{2}x$, (c) $\sec 3x$, (d) $\csc 4x$
    and state the period of each.
23. Define "amplitude of a function," and state the amplitude when the function is defined by each of the following.
    (a) $y = \cos x$ (b) $y = 3 \cos x$ (c) $y = \frac{1}{3} \cos x$
24. On the same set of axes, sketch the graphs of the three functions defined in question (23) for
    $$0 \leq x \leq 2\pi .$$
    What is the period of each function?
25. State the amplitude and phase angle relative to $\sin \theta$ of the following:
    (a) $\sin \left(\theta + \frac{\pi}{3}\right)$ (b) $3 \sin \left(\theta - \frac{\pi}{2}\right)$ (c) $2.5 \sin (\theta + 50°)$
    (d) $.375 \sin (\theta - 35°)$ (e) $7.5 \sin \left(\theta + \frac{2\pi}{3}\right)$ (f) $4 \sin (\theta - 135°)$

26. State the amplitude and period of the following:

(a) $\sin 3\theta$ (b) $2 \sin (.5\theta)$

(c) $3 \sin 2\theta$ (d) $.25 \sin (.75\theta)$

(e) $\sin \left(\frac{1}{12}\theta\right)$ (f) $7.5 \sin (6.25\theta)$

Graph the functions (b) and (c).

27. State the amplitude and phase angle relative to $\cos \theta$ of the following:

(a) $2 \cos \left(\theta - \frac{\pi}{3}\right)$ (b) $1.25 \cos (\theta + 30°)$

(c) $3 \cos \left(\theta + \frac{5\pi}{6}\right)$ (d) $\cos (\theta - 120°)$

(e) $3 \sin \theta$ (f) $1.5 \sin \left(\theta - \frac{\pi}{4}\right)$

28. State the amplitude, period, and the phase angle and shift relative to $\sin \theta$ of

(a) $2.5 \sin 3\left(\theta + \frac{\pi}{4}\right)$ (b) $\frac{1}{5} \sin \left(2\theta - \frac{\pi}{3}\right)$

(c) $3 \cos 2\theta$ (d) $3.5 \sin \frac{1}{2} (\theta - 25°)$

(e) $4 \sin (3\theta + 45°)$ (f) $2 \cos 3 (\theta - 60°)$.

29. Show that (a) $\sin \theta = 0$ for $\{\theta \mid \theta = n\pi, n \in \mathrm{I}\}$,

(b) $\sin \theta = 1$ for $\left\{\theta \mid \theta = \left(2n + \frac{1}{2}\right)\pi, n \in \mathrm{I}\right\}$,

(c) $\sin \theta = -1$ for $\left\{\theta \mid \theta = \left(2n - \frac{1}{2}\right)\pi, n \in \mathrm{I}\right\}$.

30. Show that (a) $\cos \theta = 1$ for $\{\theta \mid \theta = 2n\pi, n \in \mathrm{I}\}$,

(b) $\cos \theta = -1$ for $\{\theta \mid \theta = (2n + 1)\pi, n \in \mathrm{I}\}$,

(c) $\cos \theta = 0$ for $\left\{\theta \mid \theta = \left(n + \frac{1}{2}\right)\pi, n \in \mathrm{I}\right\}$.

31. Show that (a) $\sin \theta = \frac{1}{2}$ for $\left\{\theta \mid \theta = \left(2n + \frac{1}{6}\right)\pi, n \in I\right\}$,

(b) $\tan \theta = 1$ for $\left\{\theta \mid \theta = \left(n + \frac{1}{4}\right)\pi, n \in I\right\}$,

(c) $\cos^2\theta = \frac{3}{4}$ for $\left\{\theta \mid \theta = \left(n + \frac{1}{6}\right)\pi, n \in I\right\}$,

(d) $\sin^4\theta = 1$ for $\left\{\theta = \left(n + \frac{1}{2}\right)\pi, n \in I\right\}$,

(e) $\tan^2\theta = 1$ for $\left\{\theta \mid \theta = \left(\frac{n}{2} + \frac{1}{4}\right)\pi, n \in I\right\}$.

32. Find the minimum and maximum values of the following functions and state for which values of θ they occur.

(a) $1 + \sin \theta$

(b) $2 - 3 \cos 5\theta$

(c) $2^{\cos \theta}$

(d) $\cos (\cos \theta)$

(e) $7 + 2 \cos (\sin \theta)$

Chapter 6

TRIGONOMETRIC FUNCTIONS OF COMPOUND ANGLES

6.1. Functions of $A+B$, $A, B \in Re$

We shall develop an identity for the cosine of the sum of two real numbers (or of the sum of the measures of two angles) in terms of the cosines of the separate real numbers (or angles).

Consider the unit circle in Figure 6.1. $R(1, 0)$ is on the x-axis; P is located

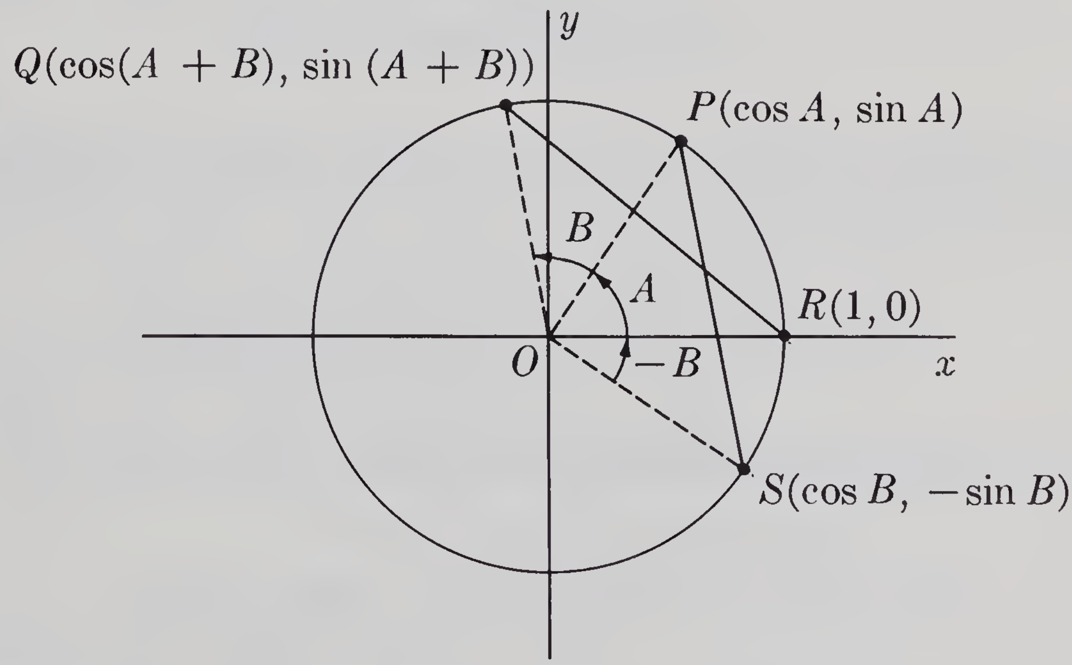


Figure 6.1

by a positive rotation from the initial ray OR through angle A, and Q by a further rotation through angle B. S is located by a negative rotation from OR through angle B. Since chords RQ and PS subtend equal angles $A + B$ at the centre of the circle,

$$RQ = PS;$$

$$\sqrt{[\cos(A+B) - 1]^2 + [\sin(A+B) - 0]^2} = \sqrt{(\cos A - \cos B)^2 + (\sin A + \sin B)^2}.$$

Squaring each member of the equation and expanding, we obtain

$$\cos^2(A+B) - 2\cos(A+B) + 1 + \sin^2(A+B)$$
$$= \cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A + 2\sin A\sin B + \sin^2 B.$$

Therefore, using the identity $\cos^2\theta + \sin^2\theta = 1$ for $\theta = A$ and $\theta = A + B$, we obtain

$$2 - 2\cos(A+B) = 2 - 2\cos A\cos B + 2\sin A\sin B\,,$$

and thus

$$\cos(A+B) = \cos A\cos B - \sin A\sin B\,.$$

Note that A and B are any two angles or real numbers.

Example 1. Without using tables, find $\cos 105°$.

Solution:

$$\begin{aligned}\cos 105° &= \cos(60° + 45°)\\ &= \cos 60°\cos 45° - \sin 60°\sin 45°\\ &= \frac{1}{2}\cdot\frac{1}{\sqrt{2}} - \frac{\sqrt{3}}{2}\cdot\frac{1}{\sqrt{2}}\\ &= \frac{\sqrt{2} - \sqrt{6}}{4}\,.\end{aligned}$$

The above formula for $\cos(A+B)$ is true for all A and B; therefore, we may replace B by $-B$.

Since

$$\cos(A+B) = \cos A\cos B - \sin A\sin B\,,$$

then

$$\cos[A+(-B)] = \cos A\cos(-B) - \sin A\sin(-B)\,.$$

Now,

$$\sin(-B) = -\sin B \text{ and } \cos(-B) = \cos B.$$

Therefore,

$$\cos(A-B) = \cos A\cos B + \sin A\sin B.$$

Example 2. Without using tables, find a value for $\cos 15°$.

Solution:

$$\begin{aligned}\cos 15° &= \cos(60° - 45°)\\ &= \cos 60°\cos 45° + \sin 60°\sin 45°\end{aligned}$$

$$= \frac{1}{2} \cdot \frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2} \cdot \frac{1}{\sqrt{2}}$$
$$= \frac{1 + \sqrt{3}}{2\sqrt{2}}$$
$$= \frac{\sqrt{2} + \sqrt{6}}{4} .$$

Example 3. Prove that $\cos (90° - \theta) = \sin \theta$.

Solution:

$$\begin{aligned} \cos (90° - \theta) &= \cos 90° \cos \theta + \sin 90° \sin \theta \\ &= 0 + \sin \theta \\ &= \sin \theta . \end{aligned}$$

Therefore,

$$\cos (90° - \theta) = \sin \theta .$$

Example 4. Prove that $\cos \theta = \sin (90° - \theta)$.

Solution: Since

$$\cos (90° - x) = \sin x ,$$

(proved in Example 3) we replace x by $90° - \theta$, and thus have

$$\begin{aligned} \cos[90° - (90° - \theta)] &= \sin (90° - \theta) , \\ \cos \theta &= \sin (90° - \theta) , \end{aligned}$$

as required.

Let us use the results of Examples 3 and 4 to develop a formula for the sine of the sum of two angles. Since it has been established that, for all x and y,

$$\cos (x - y) = \cos x \cos y + \sin x \sin y ,$$

we may replace x by $90°$ and y by $A + B$ and thus obtain

$$\begin{aligned} \cos [90° - (A + B)] &= \cos [(90° - A) - B] \\ &= \cos (90° - A) \cos B + \sin (90° - A) \sin B \\ &= \sin A \cos B + \cos A \sin B . \end{aligned}$$

But

$$\cos [90° - (A + B)] = \sin (A + B) ;$$

therefore,

$$\sin (A + B) = \sin A \cos B + \cos A \sin B .$$

Now, in this last result, let us replace B by $-B$.

$$\sin [A + (-B)] = \sin A \cos (-B) + \cos A \sin (-B) ,$$

and therefore,

$$\sin (A - B) = \sin A \cos B - \cos A \sin B .$$

Example 5. Without using tables, find the values of sin 75°.

Solution:

$$
\begin{aligned}
\sin 75^\circ &= \sin(45^\circ + 30^\circ)\,, \\
&= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\
&= \frac{1}{\sqrt{2}}\cdot\frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}}\cdot\frac{1}{2} \\
&= \frac{\sqrt{3}+1}{2\sqrt{2}} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}\,.
\end{aligned}
$$

We may obtain a formula for the tangent of the sum of two angles as follows:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cos x \neq 0\,,$$

and so if

$$x = A + B\,,$$

$$
\begin{aligned}
\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} \\
&= \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B}\,.
\end{aligned}
$$

Dividing numerator and denominator of the right member of this equality by $\cos A \cos B$, we obtain

$$
\begin{aligned}
\tan(A+B) &= \frac{\dfrac{\sin A \cos B}{\cos A \cos B} + \dfrac{\cos A \sin B}{\cos A \cos B}}{\dfrac{\cos A \cos B}{\cos A \cos B} - \dfrac{\sin A \sin B}{\cos A \cos B}} \\
&= \frac{\tan A + \tan B}{1 - \tan A \tan B}\,.
\end{aligned}
$$

Note that we need the restrictions that $\cos A \neq 0$, $\cos B \neq 0$ (why?), and $A + B$ be not an odd multiple of $\frac{\pi}{2}$ to ensure that $\tan(A + B)$ be defined.

Similarly, we may show that

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}\,.$$

Example 6. Without using tables, find the value of $\tan \frac{7}{12}\pi$.

Solution:

$$\begin{aligned}
\tan \frac{7}{12}\pi &= \tan\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\
&= \frac{\tan\frac{\pi}{3} + \tan\frac{\pi}{4}}{1 - \tan\frac{\pi}{3}\tan\frac{\pi}{4}} \\
&= \frac{\sqrt{3} + 1}{1 - \sqrt{3}} \\
&= \frac{(1 + \sqrt{3})^2}{1 - 3} \\
&= -\frac{1}{2}(4 + 2\sqrt{3}) \\
&= -(2 + \sqrt{3})\,.
\end{aligned}$$

EXERCISE 6.1

Use one of the formulae developed in Section 6.1 to find the value of each of the following.

1. $\sin 105°$
2. $\sin \frac{7\pi}{12}$
3. $\sin 210°$
4. $\tan \frac{\pi}{12}$
5. $\cos 75°$
6. $\tan 195°$

If x and y are acute angles such that $\sin x = \frac{2}{3}$ and $\cos y = \frac{5}{13}$ find the value of each of the following.

7. $\sin (x + y)$
8. $\cos (x + y)$
9. $\tan (x + y)$
10. $\cot (x + y)$
11. $\sin (x - y)$
12. $\cos (x - y)$
13. $\sin 2(x - y)$
14. $\cos 2(x - y)$
15. $\tan (x - y)$

Use the formulae of Section 6.1 to prove the following statements.

16. $\sin (180° - \theta) = \sin \theta$
17. $\sin (180° + \theta) = -\sin \theta$
18. $\cos (180° - \theta) = -\cos \theta$
19. $\cos (180° + \theta) = -\cos \theta$
20. $\tan (180° - \theta) = -\tan \theta$
21. $\tan (180° + \theta) = \tan \theta$

22. $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$

23. $\cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta$

24. $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

25. $\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$

26. Expand $\cos(90° + x) = \cos[90° - (-x)]$ to show that
$$-\sin x = \sin(-x).$$

27. Show that
$$\cos x = \cos(-x).$$

28. Show that
$$-\tan x = \tan(-x).$$

Find a value for each of the following without using tables.

29. $\cos(-105°)$

30. $\sin(-15°)$

31. $\tan(-165°)$

32. $\sin(-\frac{11}{12}\pi)$

33. $\cos(-\frac{5}{12}\pi)$

34. $\tan(-\frac{4}{3}\pi)$

Find a simpler expression for each of the following.

35. $\sin 75° \cos 45° + \cos 75° \sin 45°$

36. $\cos\left(A + \frac{\pi}{4}\right)\cos A - \sin\left(A + \frac{\pi}{4}\right)\sin A$

37. $\cos A \cos A - \sin A \sin A$

38. $\cos\left(-\frac{\pi}{4}\right)\cos\left(-\frac{\pi}{12}\right) - \sin\left(-\frac{\pi}{4}\right)\sin\left(-\frac{\pi}{12}\right)$

39. $\cos(A + B) + \cos(A - B)$

40. $\cos(A + B) - \cos(A - B)$

41. $\sin(A + B) + \sin(A - B)$

42. $\sin(A + B) - \sin(A - B)$

43. Find expressions for $\cot(A + B)$ and $\cot(A - B)$ in terms of $\cot A$ and $\cot B$.

44. If m, n, and θ are real numbers, prove that there is a real number ϕ such that
$$m\sin\theta + n\cos\theta = \sqrt{m^2 + n^2}\sin(\theta + \phi).$$

6.2 Sum and Difference Formulae

From the previous section, we know that,
$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$
and
$$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

From these equations we obtain

$$\sin(A + B) + \sin(A - B) = 2 \sin A \cos B, \tag{1}$$

and

$$\sin(A + B) - \sin(A - B) = 2 \cos A \sin B. \tag{2}$$

If we replace $A + B$ by x and $A - B$ by y, then,

$$A + B = x,$$
$$A - B = y,$$

and

$$2A = x + y,$$
$$2B = x - y.$$

Hence,

$$A = \frac{x + y}{2},$$

and

$$B = \frac{x - y}{2}.$$

Therefore, from (1)

$$\sin x + \sin y = 2 \sin \frac{x + y}{2} \cos \frac{x - y}{2}$$

and, from (2)

$$\sin x - \sin y = 2 \cos \frac{x + y}{2} \sin \frac{x - y}{2}$$

Example 1. Express $\sin 2x - \sin 4x$ in product form.

Solution:

$$\begin{aligned} \sin 2x - \sin 4x &= 2 \cos \frac{2x + 4x}{2} \sin \frac{2x - 4x}{2} \\ &= 2 \cos 3x \sin(-x) \\ &= -2 \cos 3x \sin x. \end{aligned}$$

Example 2. Calculate the value of $\sin 75° + \sin 15°$.

Solution:

$$\begin{aligned} \sin 75° + \sin 15° &= 2 \sin \frac{90°}{2} \cos \frac{60°}{2} \\ &= 2 \sin 45° \cos 30° \\ &= 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) \\ &= \frac{\sqrt{3}}{\sqrt{2}} \\ &= \frac{\sqrt{6}}{2}. \end{aligned}$$

EXERCISE 6.2

1. Using the formulae for $\cos(A + B)$ and $\cos(A - B)$, establish formulae for $\cos x + \cos y$ and $\cos x - \cos y$.

2. Express each of the following in product form.
   (a) $\sin 6x + \sin 2x$
   (b) $\cos 6x + \cos 2x$
   (c) $\sin A - \sin 2A$
   (d) $\cos A - \cos 2A$
   (e) $\sin 5\theta + \sin 3\theta$
   (f) $\cos 5\theta + \cos 3\theta$
   (g) $\sin(A + B) - \sin A$
   (h) $\cos(A + B) - \cos A$

3. Calculate the values of
   (a) $\sin 75° - \sin 15°$,
   (b) $\cos 75° - \cos 15°$,
   (c) $\sin 15° - \sin 105°$,
   (d) $\cos 15° - \cos 105°$.

6.3 Angle Between Two Lines

An interesting application of the formula for $\tan(A + B)$ lies in finding the angle between two lines.

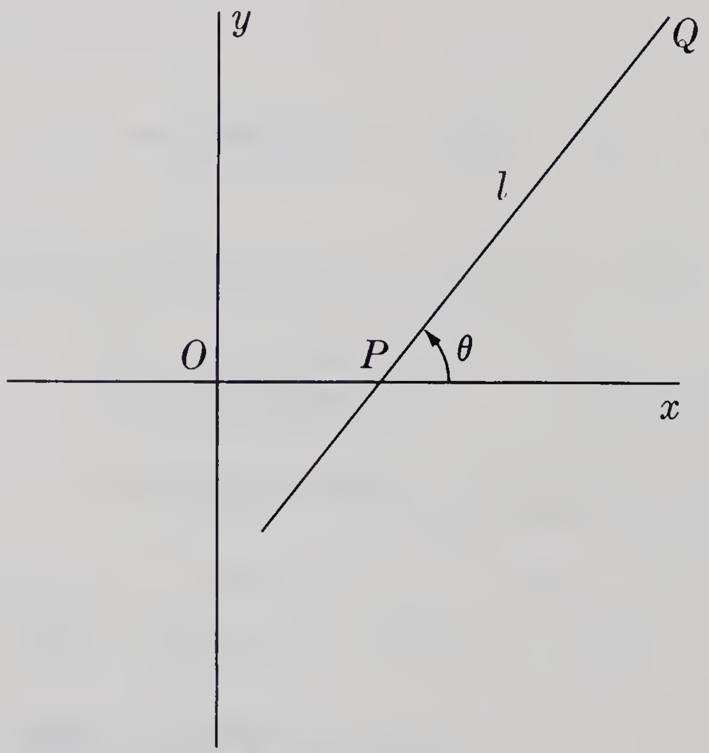


Figure 6.2

If line l intersects the x-axis at P and if Q is a point on l in the first or second quadrant, then angle xPQ, measured in a positive sense, is the angle that line l makes with the x-axis. This angle of inclination is denoted by θ in the diagram above. In Figure 6.3, l_1 and l_2 are lines whose angles of inclination are θ_1 and θ_2.

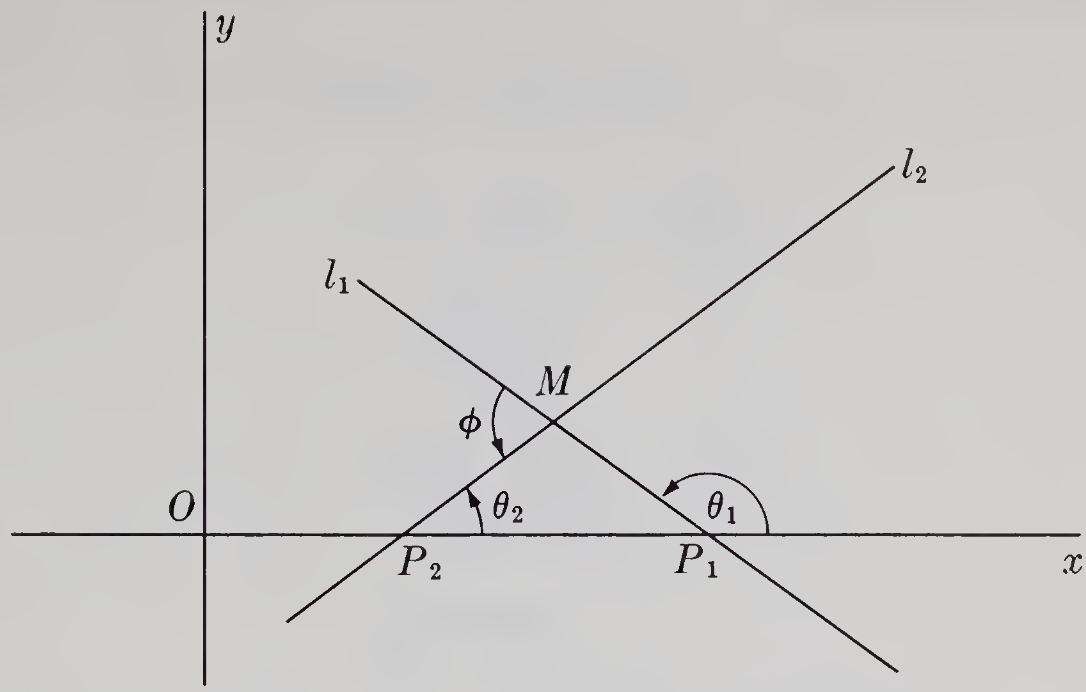


Figure 6.3

We define the angle between l_1 and l_2 as the acute angle at M, the point of intersection.

In Figure 6.3

$$\theta_1 - \theta_2 > 90° \quad \text{and} \quad \angle NMP_2 \text{ is the acute angle } \phi .$$

and, in triangle MP_1P_2,

$$\angle NMP_2 = \theta_1 - \theta_2 .$$

Hence,

$$\phi = 180 - (\theta_1 - \theta_2) .$$

and

$$\begin{aligned} \tan\phi &= \tan[180 - (\theta_1 - \theta_2)] \\ &= -\tan(\theta_1 - \theta_2) \\ &= \tan[-(\theta_1 - \theta_2)] \\ &= \tan(\theta_2 - \theta_1) \\ &= \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1} . \end{aligned}$$

But

$$\tan\theta_1 = m_1$$

and

$$\tan\theta_2 = m_2$$

where m_1 and m_2 are the slopes of l_1 and l_2.

Hence,

$$\begin{aligned} \tan\phi &= \frac{m_2 - m_1}{1 + m_2 m_1} \\ &= \frac{m_2 - m_1}{1 + m_1 m_2} \end{aligned}$$

Suppose, as in Figure 6.4,

$$\theta_1 - \theta_2 < 90^\circ \text{ and } \angle P_2MP_1 \text{ is acute.}$$

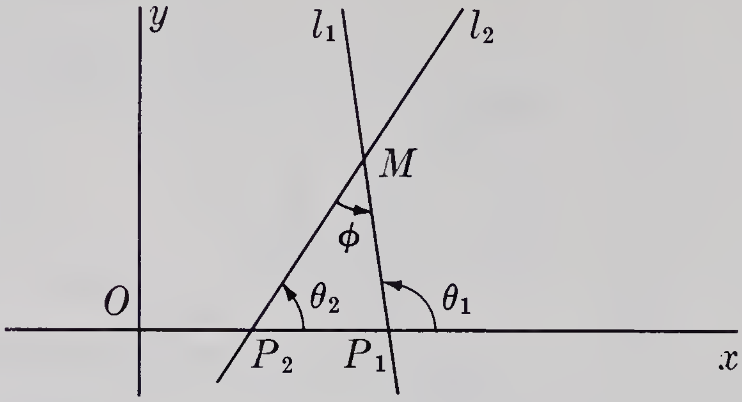


Figure 6.4

Then,

$$\theta_1 = \theta_2 + \phi\,,$$

or

$$\phi = \theta_1 - \theta_2\,,$$

and, as in the previous case,

$$\tan \phi = \tan (\theta_1 - \theta_2)$$

Using the same methods as in the previous case, we find

$$\tan \phi = \frac{m_1 - m_2}{1 + m_1m_2}\,.$$

Hence, in either case the acute angle ϕ is given by $\tan \phi = \left| \dfrac{m_1 - m_2}{1 + m_1m_2} \right|$.

The formula for $\tan \phi$ is applicable unless $m_1m_2 = -1$, that is, $m_2 = -\dfrac{1}{m_1}$.

In this case, $l_1 \perp l_2$ and $\tan \phi$ is undefined; the angle between l_1 and l_2 is 90° Thus, if $m_1m_2 = -1$, the two lines are perpendicular.

Example 1. Find the angle between the lines defined by

$$3x + y = 15\,,$$

and

$$3x - 2y + 6 = 0\,.$$

Solution: If

$$3x + y = 15 \quad \text{defines } l_1\,,$$

and

$$3x - 2y + 6 = 0 \quad \text{defines } l_2\,,$$

then

$$m_1 = -3\,,$$

and

$$m_2 = \tfrac{3}{2}\,.$$

The angle ϕ, between l_1, and l_2 is given by

$$
\begin{aligned}
\tan\phi &= \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| \\
&= \left|\frac{\frac{3}{2} - (-3)}{1 + (\frac{3}{2})(-3)}\right| \\
&= \left|\frac{\frac{9}{2}}{-\frac{7}{2}}\right| \\
&= \tfrac{9}{7} \\
&\simeq 1.29\,.
\end{aligned}
$$

Therefore,

$$\phi \simeq 52^\circ\,. \qquad \text{(tables)}$$

Example 2. Find the angle between l_1 defined by

$$x - 2y + 5 = 0\,,$$

and l_2 defined by

$$4x - 3y - 8 = 0\,.$$

Solution:

$$m_1 = \tfrac{1}{2} \quad \text{and} \quad m_2 = \tfrac{4}{3}\cdot$$

$$
\begin{aligned}
\tan\phi &= \left|\frac{m_2 - m_1}{1 + m_2 m_1}\right| \\
&= \left|\frac{\frac{4}{3} - \frac{1}{2}}{1 + (\frac{4}{3})(\frac{1}{2})}\right| \\
&= \frac{\frac{5}{6}}{\frac{10}{6}} \\
&= \tfrac{1}{2}\cdot
\end{aligned}
$$

$$\phi \simeq 27^\circ$$

The acute angle between l_1 and l_2 is approximately 27°.

EXERCISE 6.3

Find (to the nearest degree) the acute angle between the two lines defined by the following pairs of equations.

1. $3x - 2y + 8 = 0, \quad x - y - 4 = 0$
2. $5x - 8y - 10 = 0, \quad x - 3y - 5 = 0$

3. $14x - 12y - 1 = 0, \quad 3x + y - 8 = 0$

4. $4x - 8y + 17 = 0, \quad 4x + 3y + 5 = 0$

5. $3y - 4x = 2, \quad x = 7y$

6. Find the acute angle between the line through $(-2, -1)$ and $(5, 7)$, and the line through $(11, -1)$ and $(7, 4)$.

7. If the angle from a line whose slope is $\frac{1}{4}$ to a second line with greater slope m_2 is 45°, find m_2.

8. If $\phi = 30°$ and $m_2 = 2$, find m_1 (as an irrational number in its simplest form).

9. Two lines intersect at the point whose co-ordinates are $(3, 7)$. The angle between l_1 and l_2 is 45° and an equation for l_1 is
$$x + 3y - 12 = 0\,.$$
Find an equation for l_2 if its slope is positive.

10. Show that the tangent of the acute angle between the lines whose slopes are $\sqrt{2}$ and $\sqrt{3}$ is $\frac{1}{5}(4\sqrt{2} - 3\sqrt{3})$.

6.4 Formulae for sin 2θ, cos 2θ, tan 2θ

The formulae numbered (1), (2), and (3) below are three of those that we developed in Section 6.1. They permit us to develop formulae for $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, in terms of $\sin\theta$, $\cos\theta$, and $\tan\theta$ respectively,

$$\sin(A + B) = \sin A \cos B + \cos A \sin B \tag{1}$$
$$\cos(A + B) = \cos A \cos B - \sin A \sin B \tag{2}$$
$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \tag{3}$$

From (1), replacing B by A, we obtain

$$\sin(A + A) = \sin A \cos A + \cos A \sin A,$$

or

$$\sin 2A = 2 \sin A \cos A\,.$$

From (2), replacing B by A, we obtain

$$\cos(A + A) = \cos A \cos A - \sin A \sin A\,,$$

or

$$\cos 2A = \cos^2 A - \sin^2 A\,.$$

Since

$$\sin^2 A + \cos^2 A = 1,$$

we may write

$$\cos 2A = 1 - \sin^2 A - \sin^2 A\,,$$

and hence

$$\cos 2A = 1 - 2\sin^2 A\ ,$$

or

$$\cos 2A = \cos^2 A - (1 - \cos^2 A)\ ,$$

and hence

$$\cos 2A = 2\cos^2 A - 1\ .$$

From (3), replacing B by A, we obtain,

$$\tan(A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}\ ,$$

Therefore,

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}\ .$$

The formulae of this section and those of Section 6.1 allow us to prove many other identities. Consider the following examples.

Example 1. Develop a formula for $\sin 3A$ in terms of $\sin A$.

Solution:

$$\begin{aligned}\sin 3A &= \sin(A + 2A)\\ &= \sin A\cos 2A + \cos A\sin 2A\\ &= \sin A(1 - 2\sin^2 A) + \cos A(2\sin A\cos A)\\ &= \sin A - 2\sin^3 A + 2\sin A(1 - \sin^2 A)\ .\end{aligned}$$

Therefore,

$$\sin 3A = 3\sin A - 4\sin^3 A\ .$$

Example 2. Prove that, for $\theta \in Re$, $\theta \neq (2n + 1)\pi$, $n \in I$

$$\frac{\sin\theta}{1 + \cos\theta} = \tan\frac{\theta}{2}\ .$$

Solution:

$$\begin{aligned}\frac{\sin\theta}{1 + \cos\theta} &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1 + \left(2\cos^2\frac{\theta}{2} - 1\right)}\\ &= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}\cos\frac{\theta}{2}}\\ &= \tan\frac{\theta}{2}\ .\end{aligned}$$

Thus,

$$\frac{\sin\theta}{1 + \cos\theta} = \tan\frac{\theta}{2}\ .$$

Example 3 . Find a value of $\cos 22\frac{1}{2}°$ without using tables.
Solution:

$$\cos 2A = 2\cos^2 A - 1\,,$$

$$\cos 45° = 2\cos^2 22\tfrac{1}{2}° - 1\,,$$

$$2\cos^2 22\tfrac{1}{2}° = 1 + \frac{1}{\sqrt{2}}$$

$$= \frac{\sqrt{2}+1}{\sqrt{2}}$$

$$= \frac{2+\sqrt{2}}{2}\,,$$

$$\cos 22\tfrac{1}{2}° = \sqrt{\frac{2+\sqrt{2}}{4}}$$

$$= \frac{1}{2}\sqrt{2+\sqrt{2}}\,.$$

$22\frac{1}{2}°$ is a first quadrant angle; thus, $\cos 22\frac{1}{2}° > 0$.

EXERCISE 6.4

Change each of the following to an equivalent expression in terms of half the given angles.

1. $\cos 4A$
2. $\sin 4A$
3. $\tan 4A$
4. $\sin 12A$
5. $\cos 12A$
6. $\sin 10\pi$
7. $\cos \frac{\pi}{4}$
8. $\cos \frac{7\pi}{2}$
9. $\tan 240°$
10. If

$$\sin A = \tfrac{5}{13}$$

and

$$0 < A < \frac{\pi}{2}\,,$$

find $\sin 2A$, $\cos 2A$, $\tan 2A$. In which quadrant does $2A$ lie?

11. Repeat question (10), given that

$$\cos A = \frac{1}{\sqrt{5}}$$

and

$$0 < A < \frac{\pi}{2}\,.$$

12. Develop a formula for $\cos 3A$ in terms of $\cos A$.

13. If
$$\cos A = \frac{4}{5}$$
and
$$0 < A < \frac{\pi}{2},$$
find $\cos 3A$.

14. Develop formulae for $\cot 2A$ in terms of (a) $\tan A$, (b) $\cot A$.

15. If
$$\cos \theta = \tfrac{4}{5}, \qquad 0 < \theta < \frac{\pi}{2}$$
find values for $\cos \frac{\theta}{2}$ and $\sin \frac{\theta}{2}$.

16. If
$$\sin x = \tfrac{\sqrt{3}}{2},$$
find values for $\cos \frac{x}{2}$ and $\sin \frac{x}{2}$.

17. Prove
$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}.$$
Why do we use positive and negative signs?

18. Develop formulae for $\cos \frac{\theta}{2}$ and $\tan \frac{\theta}{2}$ from the formula in question (17).

Prove the following identities. State any necessary restriction on the variables involved.

19. $\cos 2x = \cos^4 x - \sin^4 x$

20. $\sqrt{2}\,\sqrt{1 - \cos 2x} = 4 \sin \frac{x}{2} \cos \frac{x}{2}$

21. $\left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2}\right)^2 = 1 - \sin \theta$

22. $\sin 2B = 1 - 2 \sin^2 \left(\frac{\pi}{4} - B\right)$

23. $\dfrac{1 + \sin y}{\cos y} = \dfrac{1 + \tan \frac{y}{2}}{1 - \tan \frac{y}{2}}$

24. $\dfrac{2}{1 + \cos A} = \sec^2 \dfrac{A}{2}$

25. $\tan \dfrac{\theta}{2} = \dfrac{1 - \cos \theta}{\sin \theta}$

26. $\tan A = \pm \sqrt{\dfrac{1 - \cos 2A}{1 + \cos 2A}}$

27. $\dfrac{\sin y \cos y}{1 - 2 \sin^2 y} = \dfrac{1}{\cot y - \tan y}$

28. $\cos \dfrac{5x}{2} \cdot \cos \dfrac{x}{2} + \sin \dfrac{5x}{2} \cdot \sin \dfrac{x}{2} = \cos 2x$

29. $\sin \dfrac{7\pi}{4} \cdot \cos \dfrac{3\pi}{4} - \cos \dfrac{7\pi}{4} \cdot \sin \dfrac{3\pi}{4} = 0$

30. $\cos x + \cos 2x + \cos 3x = \cos 2x(1 + 2 \cos x)$

31. $2(\cos^4 A + \sin^4 A) = 1 + \cos^2 2A$ 32. $\sin 3y + \sin y = 2 \sin 2y \cos y$

33. Explain the restriction on θ in Example 2 of Section 6.4.

6.5 Addition of Sine and Cosine Functions

If we wish to construct the graph of $y = \sin \theta + \cos \theta$, we could construct a table of values and, from this, draw the graph. However,

$$\begin{aligned}
\sin \theta + \cos \theta &= \sin \theta + \sin (90° - \theta) \\
&= 2 \sin \frac{90°}{2} \cos \frac{2\theta - 90°}{2} \\
&= 2 \sin 45° \cos (\theta - 45°) \\
&= 2(\tfrac{1}{\sqrt{2}}) \cos (45° - \theta) \\
&= \sqrt{2} \sin (90° - 45° + \theta) \\
&= \sqrt{2} \sin (\theta + 45°) .
\end{aligned}$$

We note that the graph is sinusoidal; that is, it has the same shape as the graph of the sine function. The amplitude is $\sqrt{2}$, the period is 2π, and the phase angle is 45° or $\frac{\pi}{4}$.

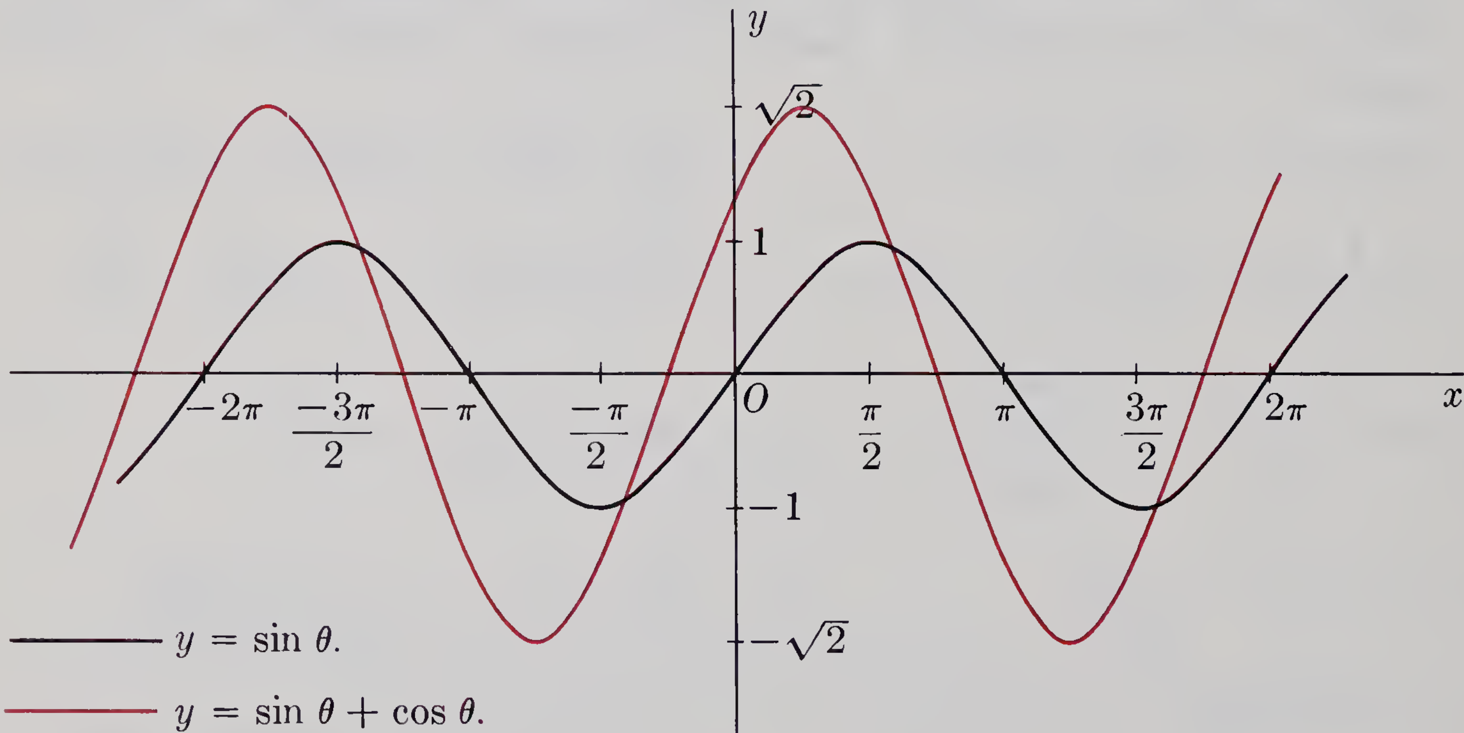


Figure 6.5

A more general function formed from the sine and cosine functions is the function determined by $y = a \sin \theta + b \cos \theta$. The example above is the special case obtained when $a = b = 1$.

From the special case we see that

$$\begin{aligned}\sqrt{2} \sin (\theta + 45°) &= \sqrt{2} (\sin \theta \cos 45° + \cos \theta \sin 45°) \\ &= (\sqrt{2} \cos 45°) \sin \theta + (\sqrt{2} \sin 45°) \cos \theta . \\ &= \sin \theta + \cos \theta .\end{aligned}$$

This suggests that in the general case we might try to find two numbers k and ϕ such that

$$\begin{aligned}a \sin \theta + b \cos \theta &= k \cos \phi \sin \theta + k \sin \phi \cos \theta \\ &= k \sin (\theta + \phi) .\end{aligned}$$

If these numbers exist, we must have

$$a = k \cos \phi \text{ and } b = k \sin \phi .$$

By squaring and adding, we obtain

$$\begin{aligned}a^2 + b^2 &= k^2 \cos^2 \phi + k^2 \sin^2 \phi \\ &= k^2 ,\end{aligned}$$

and by elimination of k we obtain

$$\begin{aligned}\frac{b}{a} &= \frac{\sin \phi}{\cos \phi} \\ &= \tan \phi .\end{aligned}$$

Example 1. Find k and ϕ so that

$$4 \sin \theta + 3 \cos \theta = k \sin (\theta + \phi) .$$

Solution:

$$\begin{aligned}k^2 &= 4^2 + 3^2 \\ &= 25\end{aligned}$$

Therefore,

$$k = 5 ,$$

and

$$\tan \phi = \frac{3}{4}$$

Hence,

$$\phi \simeq 37°$$

Therefore,

$$4 \sin \theta + 3 \cos \theta = 5 \sin (\theta + 37°) .$$

In Figure 6.6(a), we show the graphs of $y = 4 \sin \theta$ and $y = 3 \cos \theta$ and in Figure 6.6(b) we show the graph of

$$y = 4 \sin \theta + 3 \cos \theta$$
$$= 5 \sin (\theta + 37°) .$$

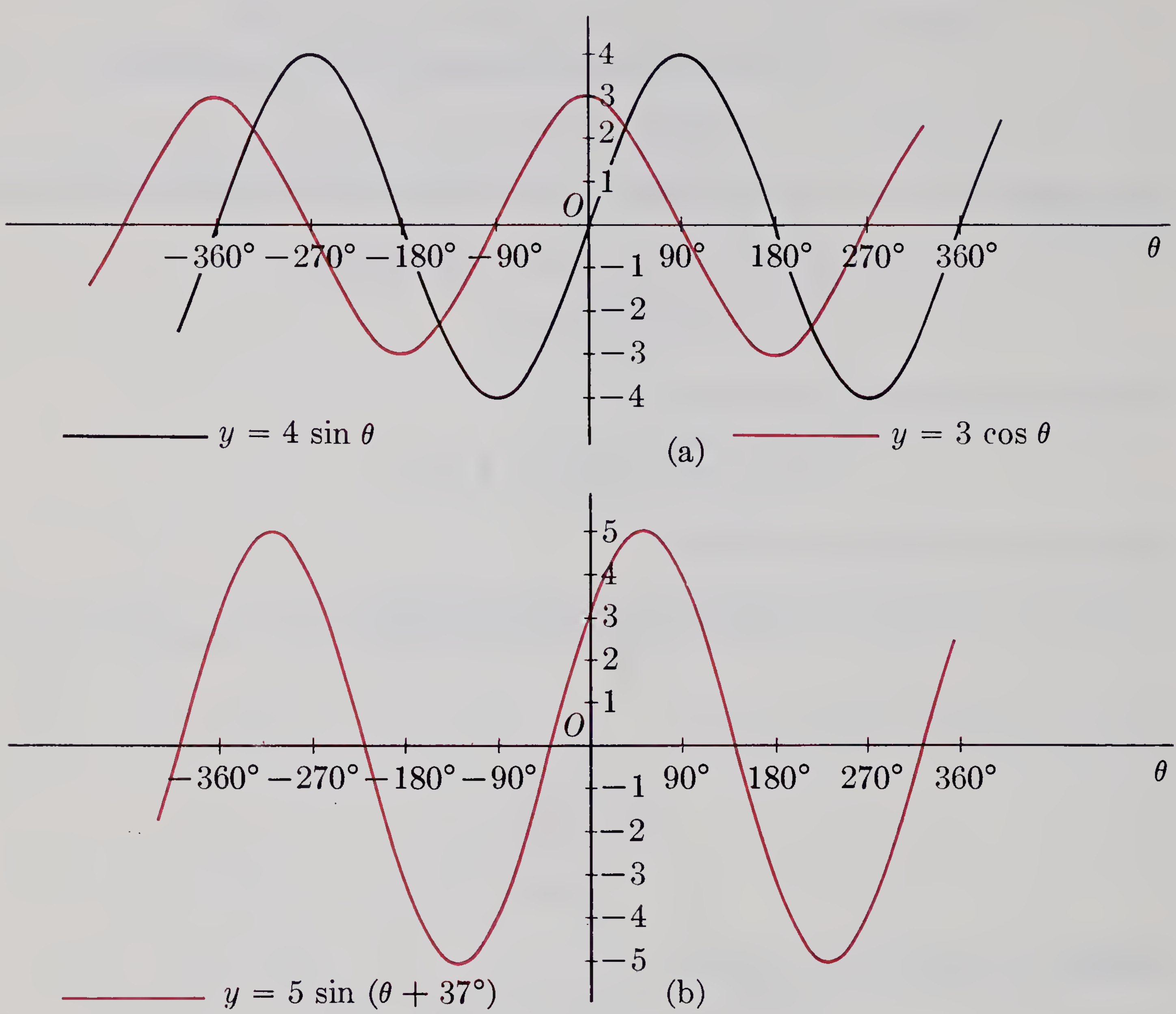


Figure 6.6

EXERCISE 6.5

1. Find the period and amplitude of the following functions.
   (a) $\sin \theta + \sqrt{3} \cos \theta$ (b) $-3 \sin \theta + 4 \cos \theta$
   (c) $5 \sin x + 12 \cos x$ (d) $2 \sin x - \cos x$
   (e) $\sin 2\theta + \sqrt{3} \sin 2\theta$ (f) $\sin 3\pi x + 2 \cos 3\pi x$

2. In the preceding question, find the phase angle of the functions in (a) to (d) with respect to both sine and cosine functions.

3. In question (1), sketch the graphs of parts (a) to (f).

Chapter Summary

Compound angle formulae:

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2x = 2 \sin x \cos x$$
$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x \\ &= 2\cos^2 x - 1 \\ &= 1 - 2\sin^2 x\end{aligned}$$
$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

Sum and difference formulae:

$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2}$$
$$\sin x - \sin y = 2 \cos \frac{x+y}{2} \sin \frac{x-y}{2}$$
$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}$$
$$\cos x - \cos y = -2 \sin \frac{x+y}{2} \sin \frac{x-y}{2}$$

If a line l_1 has slope m_1 and a line l_2 has slope m_2, then the acute angle ϕ between l_1 and l_2 is given by

$$\tan \phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right| .$$

The equation

$$a \sin \theta + b \cos \theta = k \sin(\theta + \phi),$$

where

$$k^2 = a^2 + b^2,$$

and

$$\tan \phi = \frac{b}{a}.$$

REVIEW EXERCISE 6

1. Find the acute angle between the line whose equation is

$$x - 4y - 2 = 0,$$

and the line whose equation is

$$4x + 4y - 17 = 0.$$

2. Find the acute angle between the line that passes through $(7, -1)$ and $(4, 8)$ and the line that has a slope of $1/2$ and a y-intercept of 5.

3. Show that the tangent of one of the angles between the lines whose slopes are $\sqrt{3}$ and $1/\sqrt{3}$ is $-1/\sqrt{3}$.

4. Find the value of $\sin 120°$ by expanding $\sin 2(60°)$.

Prove the following identities.

5. $\sin 5\theta \cos \theta + \cos 5\theta \sin \theta = \sin 6\theta$

6. $\cos \dfrac{7\pi}{2} \cos \dfrac{3\pi}{2} + \sin \dfrac{7\pi}{2} \sin \dfrac{3\pi}{2} = 1$

7. $\cos^2 x + \cos^2 y - 1 = \cos (x + y) \cos (x - y)$

8. $\cos (m + n) + \cos (m - n) = 2 \cos m \cos n$

9. $\sin (m + n) + \sin (m - n) = 2 \sin m \cos n$

10. $\tan (x + y + z) = \dfrac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - \tan x \tan y - \tan x \tan z - \tan y \tan z}$.

11. $\cos (\pi + \theta) + \cos (\pi - \theta) = -2 \cos \theta$

12. $\dfrac{\sin 2A}{\sin A} - \dfrac{\cos 2A}{\cos A} = \dfrac{1}{\cos A}$

13. $\cos 2x = \dfrac{1 - \tan^2 x}{1 + \tan^2 x}$

14. $\sin (\pi + \theta) + \cos \left(\dfrac{\pi}{2} - \theta\right) + \tan \left(\dfrac{\pi}{2} + \theta\right) = -\cot \theta$

15. $\sin 3A - \sin A = 2 \sin A - 4 \sin^3 A$.

16. $\cos 3A + \cos A = 4 \cos^3 A - 2 \cos A$.

17. State the period, amplitude, and phase angle, and draw the graphs of each of the following.
    (a) $\sqrt{3} \sin \theta + \cos \theta$.
    (b) $\sqrt{3} \sin \theta - \cos \theta$.
    (c) $4 \sin \theta - 3 \cos \theta$.
    (d) $2 \sin \theta + 2 \cos \theta$.

Chapter 7

TRANSLATIONS IN THE PLANE

7.1. The Mapping (x, y) $\quad$ $(x + h,\ y + k)$

In a first study of a function as a mapping, one of the simplest mappings would be

$$x \to x + 1\,, \qquad (x \in Re)\,.$$

For clarity, this mapping is often represented geometrically on two number lines as follows:

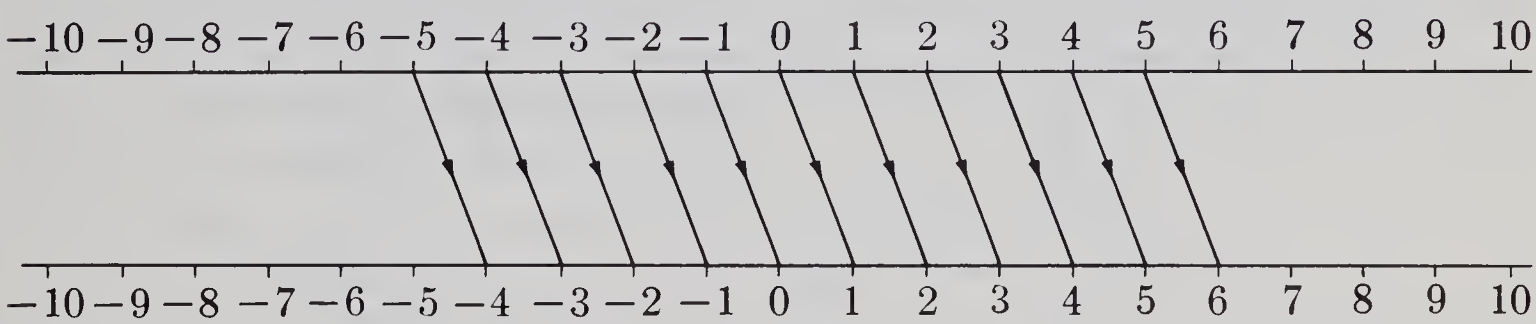


Figure 7.1

In Figure 7.1, the line segments with arrows illustrate the mapping

$$x \to x + 1, \qquad x \in I, \;\; -5 \leq x \leq +5\,.$$

If we tried to draw all the arrows for $x \to x + 1$ in the domain $x \in Re$, we could not exhibit the arrows separately.

The mapping above,

$$x \to x + 1 \qquad x \in I, \;\; -5 \leq x \leq +5\,,$$

as illustrated, maps points on one number line onto points on a second number line. If we now consider points in a region of the Cartesian plane, say (x, y), where $x, y \in I$, $-5 \leq x \leq +5$ and $-5 \leq y \leq +5$, we can map this set of points onto another set of points in the plane by means of a mapping such as

$$x \to x + 1, \;\; y \to y - 1\,.$$

In practice, for clarity, we often draw two Cartesian planes, or, more precisely, portions of them, on the same sheet of paper, just as we used two number lines in Figure 7.1.

Thus, we illustrate the mapping

$$x \to x+1, \quad y \to y-1, \quad x, y \in I, \quad -5 \leq x \leq 5, \quad -5 \leq y \leq 5,$$

in Figure 7.2.

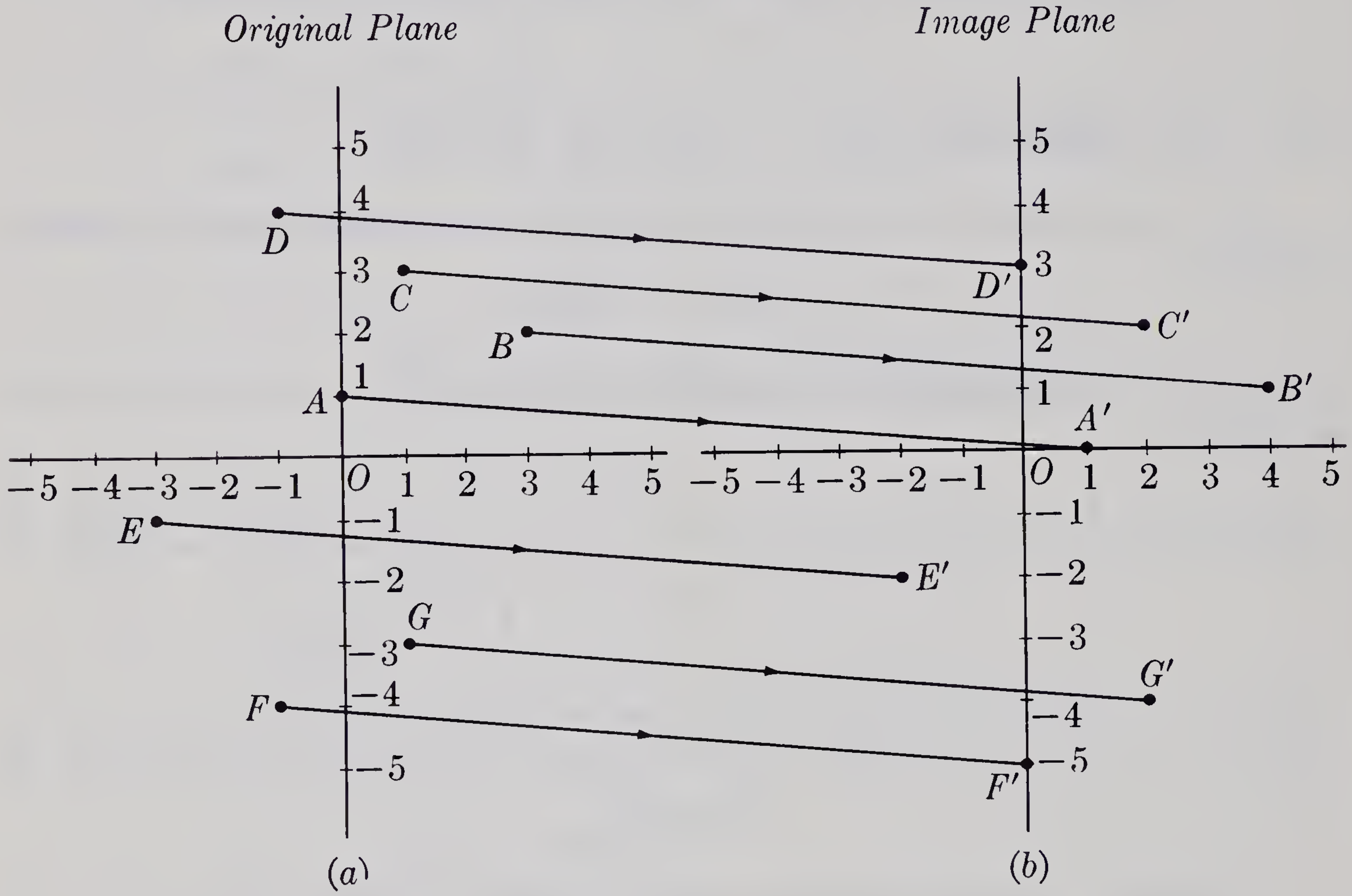


Figure 7.2

We have inserted only a few of the line segments with arrows that join corresponding points in the two planes. The diagram becomes much too confusing if we try to insert too many such line segments.

Using the terminology that we developed in Chapter 2 for functional mappings in one variable, we say that A' is the *image* of A under the mapping

$$x \to x+1, \quad y \to y-1;$$

similarly, B' is the image of B, and so on. To each point in the first Cartesian plane, (a), there corresponds a unique point in the second Cartesian plane, (b).

If we return to the consideration of the single-variable mapping

$$x \to x+1, \qquad x \in I,$$

we see that it maps the set of integers onto the set of integers. In this sense, it maps the points given by integers on a number line onto the points given by integers on the same number line. This mapping is shown in Figure 7.3, where it is illustrated by four arrows.

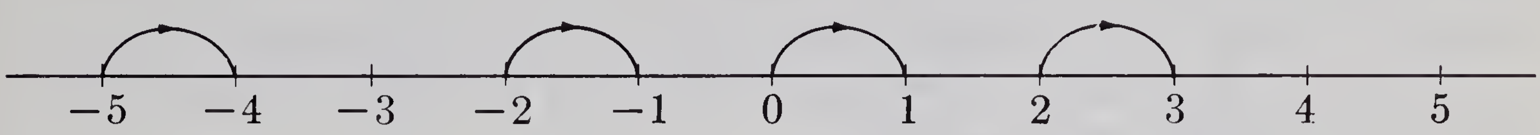


Figure 7.3

In the same sense, the two dimensional mapping

$$x \to x + 1, \quad y \to y - 1, \quad x, y \in I,$$

maps the set of ordered pairs of integers onto the set of ordered pairs of integers. Thus, the mapping maps the points given by ordered pairs of integers in a Cartesian plane onto the points given by ordered pairs of integers in the same Cartesian plane. This mapping is shown in Figure 7.4, where we have illustrated the mapping of some particular points by arrows.

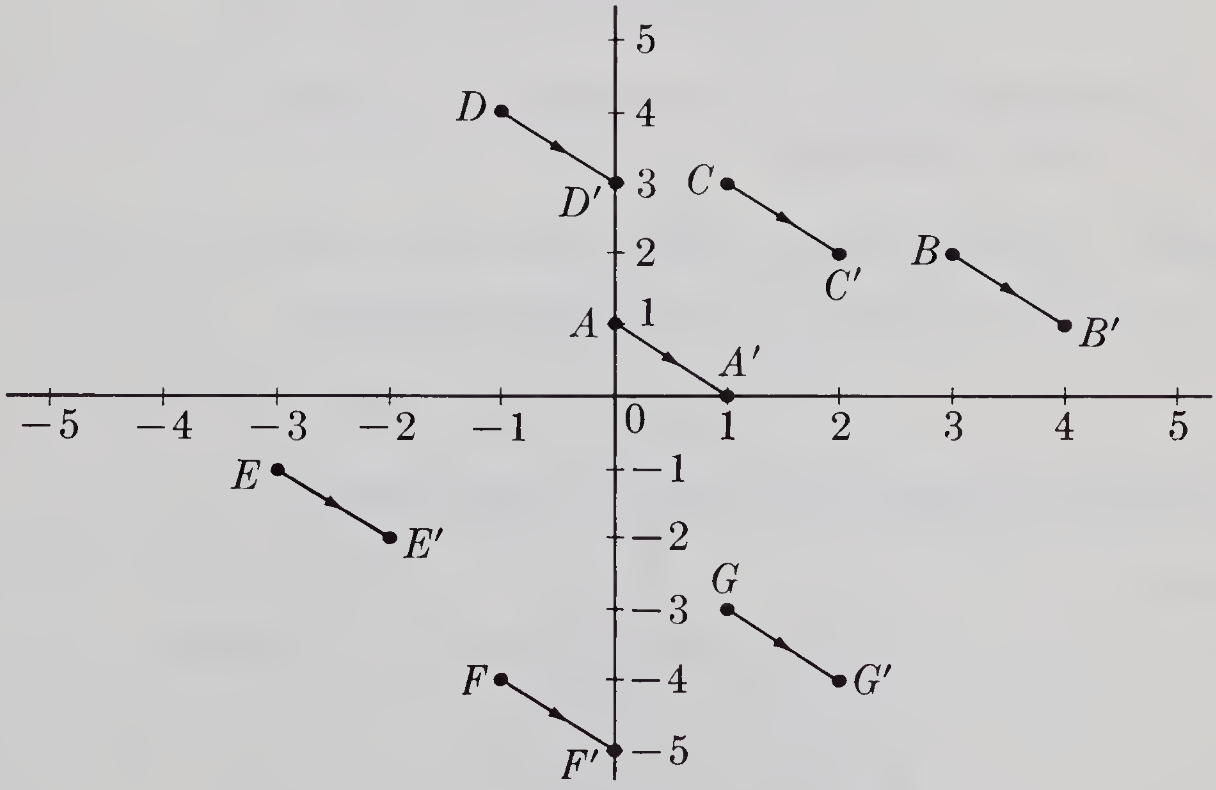


Figure 7.4

The decision on whether to use a two-plane or a single-plane representation is usually made on the basis of which diagram illustrates the situation more clearly.

If we examine Figure 7.4, we see that all the arrows connecting the original points with their images are parallel line segments of the same length and direction. Such a mapping is called a *uniform translation* or, more simply, a *translation*. If the original set of points and the axes are drawn on a sheet of rigid transparent plastic, we see that a movement of the sheet by the translation indicated by any one of the arrows, and without any rotation, will move all the original points into coincidence with their corresponding image points. Note that we are actually moving a "model" of the mathematical concept, not the mathematical points.

The above conclusions, reached on the basis of a few points, remain valid for all points in the case of the mapping

$$x \rightarrow x + 1, \quad y \rightarrow y - 1, \quad x \in Re\,.$$

If we use the two plane representation, we see that to any point in the *original plane* (the first plane) there corresponds a unique point in the *image plane* (the second plane). If we use the single plane representation, we see that each point in the plane has a unique image point in the plane. In more algebraic terms, we may say that the ordered pair (x, y) is transformed onto its image $(x + 1, y - 1)$ by the mapping

$$(x, y) \rightarrow (x + 1, y - 1) \qquad x, y \in Re\,.$$

DEFINITION. The mapping

$$(x, y) \rightarrow (x + h, y + k), \qquad x, y \in Re,$$

maps a given ordered pair (x, y) uniquely onto its image $(x + h, y + k)$. Such a mapping is called a *translation*.

Example 1. Find the images of the points $A(2, 1)$, $B(4, 3)$, $C(-2, 5)$, $D(-6, 1)$, $E(-2, -4)$, $F(0, -9)$, $G(3, -4)$ under the translation

$$(x, y) \rightarrow (x - 2, y + 3)$$

and showing the corresponding points in sketch graphs.

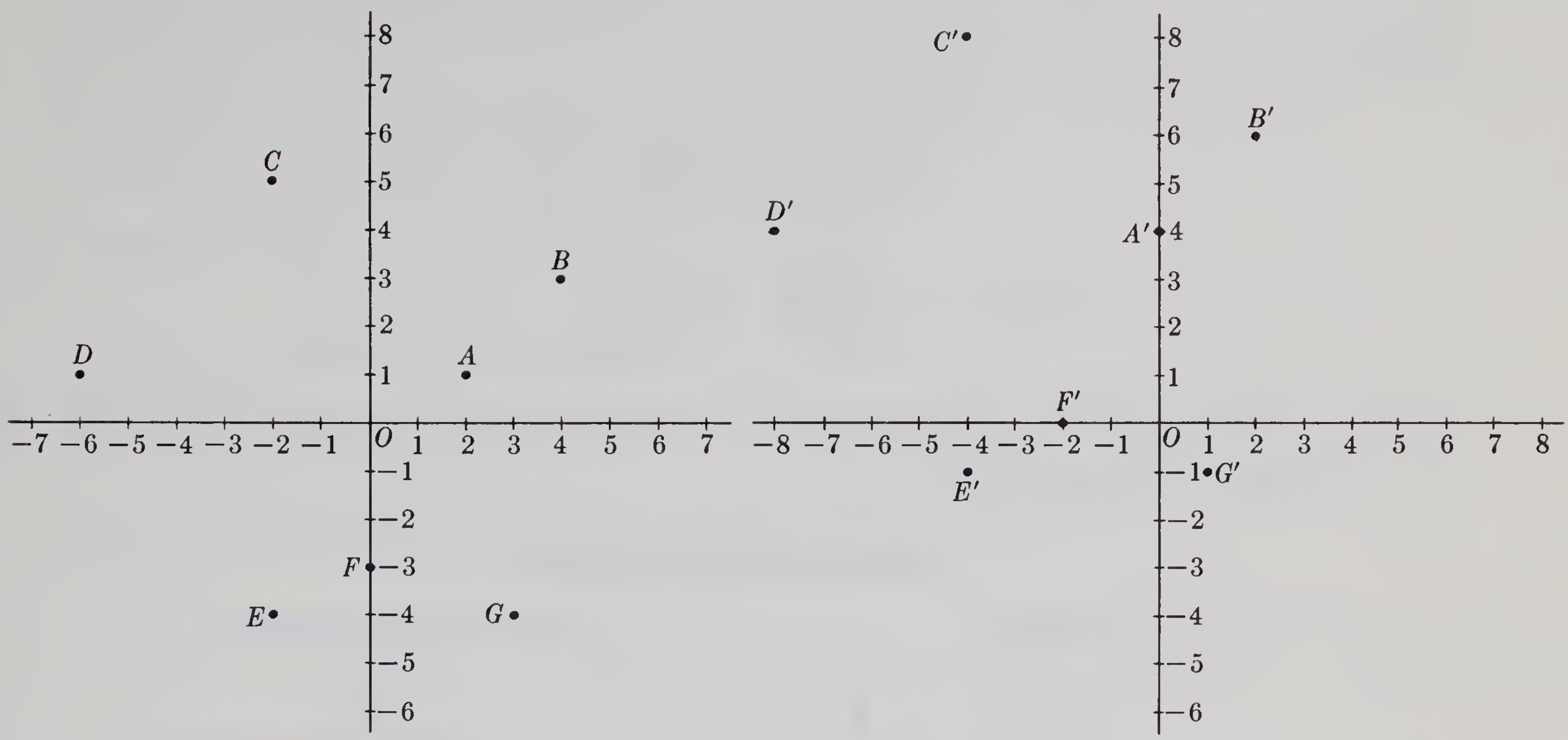
Solution:

$$\begin{aligned}
A(2, 1) &\rightarrow A'(2 - 2, 1 + 3) = A'(0, 4)\\
B(4, 3) &\rightarrow B'(2, 6)\\
C(-2, 5) &\rightarrow C'(-4, 8)\\
D(-6, 1) &\rightarrow D'(-8, 4)\\
E(-2, -4) &\rightarrow E'(-4, -1)\\
F(0, -3) &\rightarrow F'(-2, 0)\\
G(3, -4) &\rightarrow G'(1, -1)
\end{aligned}$$

We may use a two-plane or a single-plane representation.

Two-Plane Representation

Original Plane *Image Plane*



(a) (b)

Single-Plane Representation

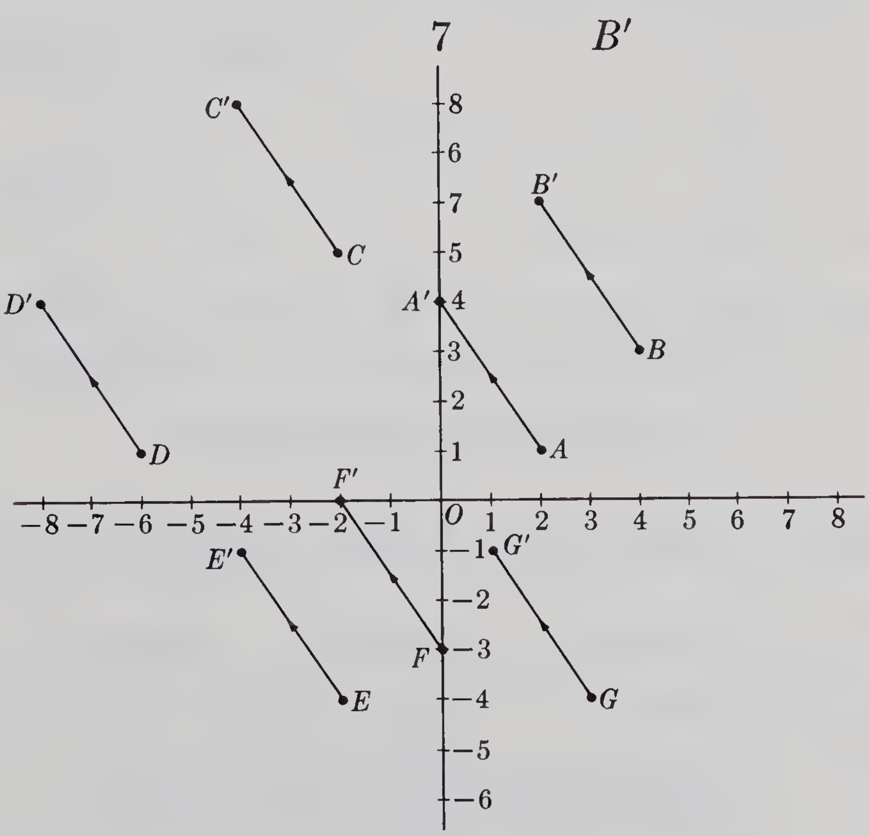


Figure 7.5

Example 2. Find images of $A(-3, -2)$, $B(-1, 0)$, $C(0, 1)$, $D(2, 3)$, $E(4, 5)$, under the translation

$$(x, y) \to (x + 1,\ \ y - 2)\ .$$

Sketch the corresponding points.

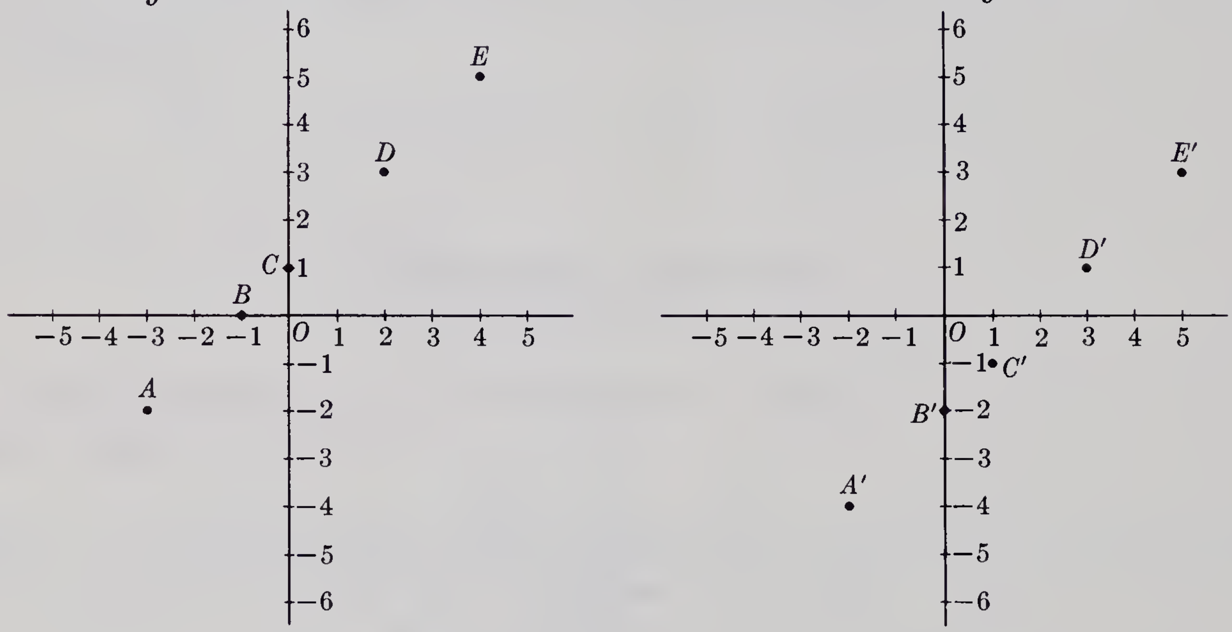
Solution:

$$\begin{aligned} A(-3, -2) &\to A'(-2, -4)\ , \\ B(-1, 0) &\to B'(0, -2)\ , \\ C(0, 1) &\to C'(1, -1)\ , \\ D(2, 3) &\to D'(3, 1)\ , \\ E(4, 5) &\to E'(5, 3) \end{aligned}$$

We note that the points $ABCDE$ lie on a line and that the points $A'B'C'D'E'$ lie on a line with the same slope.

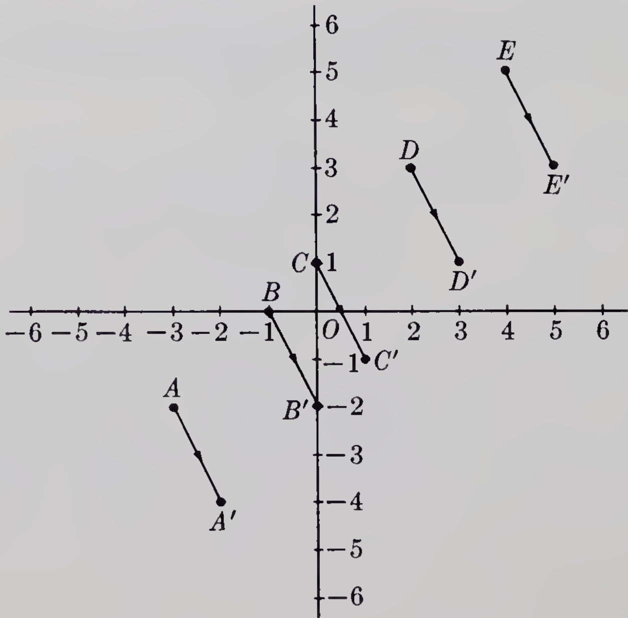
Two-Plane Representation

Original Plane *Image Plane*



(a) (b)

Single-Plane Representation



EXERCISE 7.1

1. Find the images of the points

$$A(3, -5), \quad B(-4, 3), \quad C(-2, -5), \quad D(3, 2)$$

under the following translations.

(a) $(x, y) \to (x + 1,\ y + 2)$ (b) $(x, y) \to (x - 2,\ y + 4)$

(c) $(x, y) \to (x + 3,\ y - 2)$ (d) $(x, y) \to (x - 3,\ y - 5)$

Show the corresponding points on sketch graphs.

2. Find the images of the points

$$A(-2, -1), \quad B(0, 2), \quad C(2, 5), \quad D(6, 11)$$

under the following translations.

(a) $(x, y) \to (x - 1,\ y + 1)$ (b) $(x, y) \to (x - 2,\ y - 3)$

(c) $(x, y) \to (x + 2,\ y - 3)$ (d) $(x, y) \to (x + 4,\ y + 6)$

Show the corresponding points on sketch graphs.

3. Find the images of the points

$$A(3, -2), \quad B(4, 1), \quad C(-2, 2), \quad D(-3, -1)$$

under the translations

(a) $(x, y) \to (x - 4, y - 1)$, (b) $(x, y) \to (x + 3, y + 1)$.

Sketch the graphs of the points. What geometric figure is formed by $ABCD$? What geometric figure is formed by the corresponding sets of image points?

4. Find the images of the points

$$A(1, 2), \quad B(-2, 4), \quad C(-1, -2)$$

under the translations

(a) $(x, y) \to (x - 1, y - 2)$, (b) $(x, y) \to (x + 1, y + 2)$.

Sketch the graphs of the points. What geometric figures are formed by ABC and by the image points? Do these figures appear to be congruent?

7.2. Invariance of Length and Angle under a Translation

If we consider the graphs in Example 1 of the previous section, we see that the two figures $ABCDEFG$ and $A'B'C'D'E'F'G'$ appear to be congruent. If they are, then

$$AB = A'B', \quad BC = B'C', \text{ etc.,}$$

and

$$\angle ABC = \angle A'B'C', \quad \angle BCD = \angle B'C'D', \text{ etc.}$$

We now investigate this question more precisely.

Example 1. $A(2, 1)$, $B(4, 3)$, $C(-2, 5)$ have images $A'(0, 4)$, $B'(2, 6)$, $C'(-4, 8)$ under the translation

$$(x, y) \to (x - 2, y + 3) .$$

Show that $AB = A'B'$, $BC = B'C'$, $CA = C'A'$ and $\angle ABC = \angle A'B'C'$.

Solution:

$$\begin{aligned} AB^2 &= (4 - 2)^2 + (3 - 1)^2 \\ &= 2^2 + 2^2 \\ &= 8 , \end{aligned}$$

and

$$\begin{aligned} A'B'^2 &= (2 - 0)^2 + (6 - 4)^2 \\ &= 2^2 + 2^2 \\ &= 8 , \end{aligned}$$

Thus,

$$AB^2 = A'B'^2;$$

therefore,

$$AB = A'B' .$$

Similarly,

$$\begin{aligned} BC^2 &= 6^2 + 2^2 = 40 , \\ B'C'^2 &= 6^2 + 2^2 = 40 . \\ BC &= B'C' , \end{aligned}$$

and

$$\begin{aligned} CA^2 &= 4^2 + 4^2 = 32 , \\ C'A'^2 &= 4^2 + 4^2 = 32 . \end{aligned}$$

Therefore,

$$CA = C'A' ,$$

$$\text{Slope of } AB = m_1 = \tfrac{2}{2} = 1 .$$

$$\text{Slope of } BC = m_2 = -\tfrac{2}{6} = -\tfrac{1}{3} .$$

Hence,

$$\tan (\angle ABC) = \frac{1 + \frac{1}{3}}{1 + 1(-\frac{1}{3})} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2 .$$

$$\text{Slope of } A'B' = m_1' = \tfrac{2}{2} = 1 .$$

$$\text{Slope of } B'C' = m_2' = \tfrac{2}{-6} = -\tfrac{1}{3} .$$

$$\tan (\angle A'B'C') = \frac{1 + \frac{1}{3}}{1 + 1(-\frac{1}{3})} = \frac{\frac{4}{3}}{\frac{2}{3}} = 2 .$$

Therefore,

$$\angle ABC = \angle A'B'C' .$$

Thus we have established for this translation that the length of a typical line segment is unchanged and so is the angle between two typical line segments. Mathematicians use the word *invariant* to describe this changelessness.

We note that, in fact, AB is equal to $A'B'$ and AB is parallel to $A'B'$. As BC is parallel to $B'C'$, the angle between AB and BC must be equal to the angle between $A'B'$ and $B'C'$.

In general, if the translation is

$$(x, y) \rightarrow (x + h, y + k) ,$$

then any line segment PQ and its image $P'Q'$ are equal and parallel. Let P be the point (a, b) and Q the point (c, d); then P' is $(a + h, b + k)$ and $P'Q'$ is $(c + h, d + k)$.

$$\begin{aligned} PQ^2 &= (c - a)^2 + (d - b)^2 \\ &= [(c + h) - (a + h)]^2 + [(d + k) - (b + k)]^2 \\ &= P'Q'^2 ; \end{aligned}$$

therefore,

$$PQ = P'Q' .$$

$$\begin{aligned} \text{Slope of } PQ &= \frac{d - b}{c - a} \\ &= \frac{(d + k) - (b + k)}{(c + h) - (a + h)} \\ &= \text{slope of } P'Q'. \end{aligned}$$

Thus, we have established that the length of any line segment is invariant under a translation and that the slope of any line is also an invariant under a translation.

If we consider a third point $C(e, f)$, we can show in the same way that

$$BC = B'C' \quad \text{and} \quad CA = C'A' .$$

Thus,

$$\triangle ABC \equiv \triangle A'B'C' ,$$

and from this

$$\angle ABC = \angle A'B'C' .$$

The angle between any two lines is therefore an invariant under a translation. We could also show this invariance by calculating the slope of BC and $B'C'$ and then finding the angle between AB and BC, and $A'B'$ and $B'C'$.

The following example illustrates the effect of these invariant properties for a simple polygon under translation.

Example 2. $\triangle ABC$, with vertices $A(-1,0)$, $B(1,0)$, $C(0,\sqrt{3})$, is equilateral. Show that the mapping

$$(x, y) \to (x + \sqrt{2},\ y - \sqrt{5})$$

produces an image triangle $A'B'C'$ that is congruent to $\triangle ABC$ and therefore equilateral. Show both triangles on a single plane representation.

Solution:

$A \to A'$ is

$$(-1,0) \to (-1 + \sqrt{2},\ -\sqrt{5})\,;$$

$B \to B'$ is

$$(1,0) \to (1 + \sqrt{2},\ -\sqrt{5})\,;$$

and $C \to C'$ is

$$(0, \sqrt{3}) \to (\sqrt{2}, \sqrt{3} - \sqrt{5})\,.$$

$$AB^2 = [1 - (-1)]^2\,;$$

$$AB = 2\,.$$

Hence,

$$A'B'^2 = [(+1 + \sqrt{2}) - (-1 + \sqrt{2})]^2 + [(-\sqrt{5}) - (-\sqrt{5})]^2$$

$$= 2^2\,,$$

and

$$A'B' = 2\,.$$

Thus,

$$AB = A'B'$$

$$BC^2 = (0 - 1)^2 + (\sqrt{3} - 0)^2$$

$$= 1 + 3\,,$$

and

$$BC = 2\,.$$

$$B'C'^2 = [\sqrt{2} - (1 + \sqrt{2})]^2 + [\sqrt{3} - \sqrt{5}\ - (-\sqrt{5})]^2$$

$$= 1 + 3\,,$$

and

$$B'C' = 2\,.$$

Hence,

$$BC = B'C'\,.$$

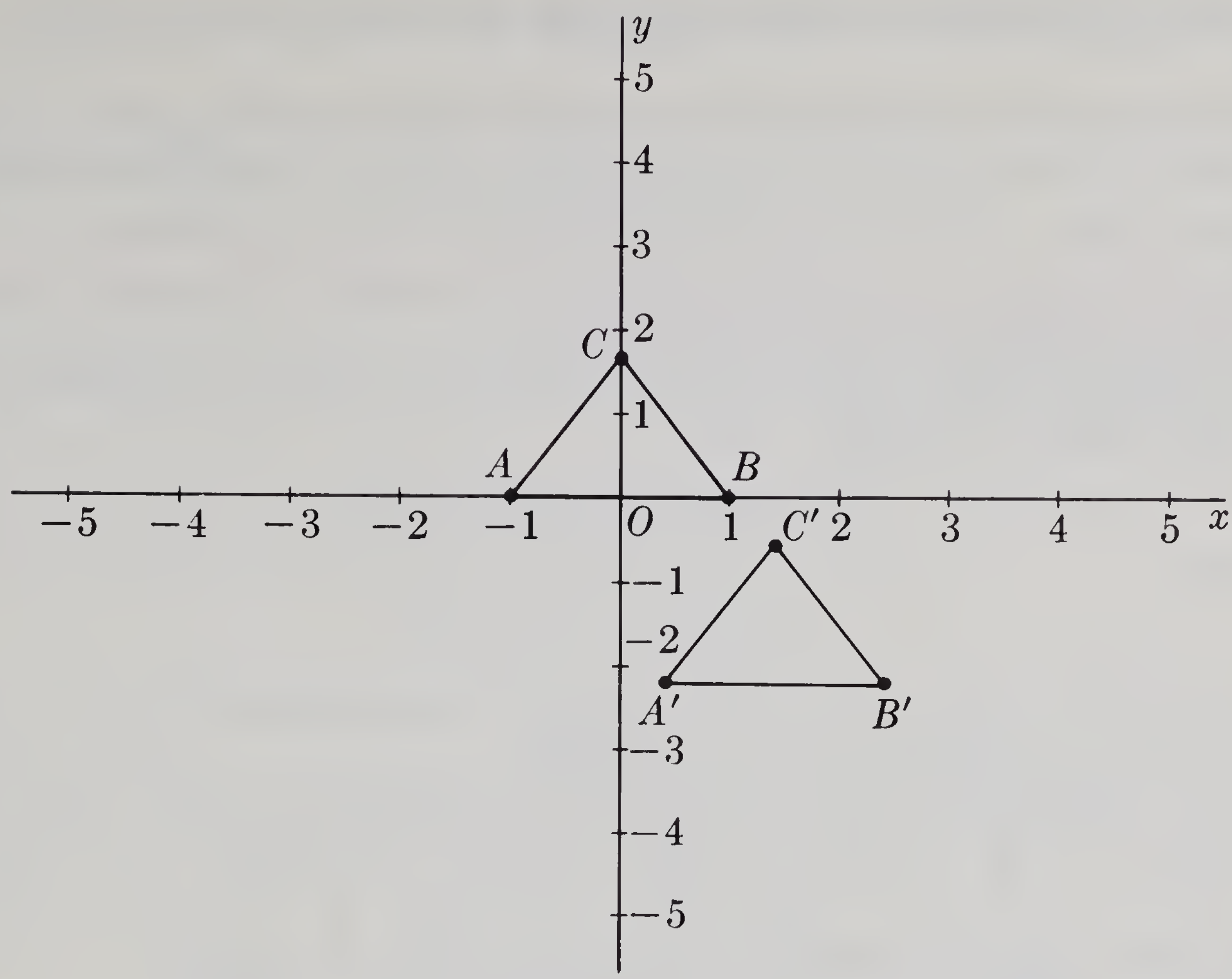
Similarly,

$$CA = C'A'\,.$$

Therefore,

$$\triangle ABC \equiv \triangle A'B'C'$$

and $\triangle ABC$ is equilateral; hence, $\triangle A'B'C'$ is equilateral.



EXERCISE 7.2

1. Find the lengths of the line segments AB, BC, CA if A is $(2, 1)$, B is $(6, 1)$, and C is $(6, 4)$. If A', B', C' are the image points of A, B, C, under the translation
$$(x, y) \to (x - 2,\ y - 1)\ ,$$
find the lengths of $A'B'$, $B'C'$, $C'A'$. Verify that
$$AB = A'B', \qquad BC = B'C', \qquad CA = C'A'.$$

2. In question (1), find the slope of AC and show that $A'C'$ has the same slope.

3. $A(0, 0)$, $B(2, 1)$, $C(3, 5)$ are three points, and A', B', C', are their respective images under
$$(x, y) \to (x - 3,\ y - 5)\ .$$
Verify that
$$\angle BAC = \angle B'A'C'$$
and that
$$\angle ACB = \angle A'C'B'\ .$$

4. Find and graph the image points of $A(2, 1)$, $B(0, 3)$, and $C(-1, 2)$ under the transformation
$$(x, y) \to (x + 2,\ y - 1)\ .$$
Show that the two triangles are congruent.

7.3. The Equation of the Image of the Line $y = mx + b$

So far, we have considered the mapping of isolated points and line segments in a given Cartesian plane and have plotted the results of a translation in an image Cartesian plane. In order to distinguish easily between co-ordinates in the two planes, we shall use (x, y) for the co-ordinates of a point in the original plane and (u, v) for the image point in the image plane.

We first note that, if (u, v) is the image point of (x, y), the general translation mapping may be written

$$(x, y) \rightarrow (x + h,\ y + k) = (u, v)\ ,$$

and hence

$$u = x + h, \qquad v = y + k\ .$$

These equations express the relation between the co-ordinates in the original plane and the co-ordinates in the image plane.

In the original plane, the x-axis is the line on which y is equal to zero; that is, its equation is $y = 0$. In the new plane, therefore, it is mapped onto the line with the equation $v = k$ in the image plane. Similarly, the y-axis is $x = 0$ in the original plane, and it is mapped onto the line with the equation $u = h$ in the image plane. This mapping of the axes is illustrated in Figure 7.6.

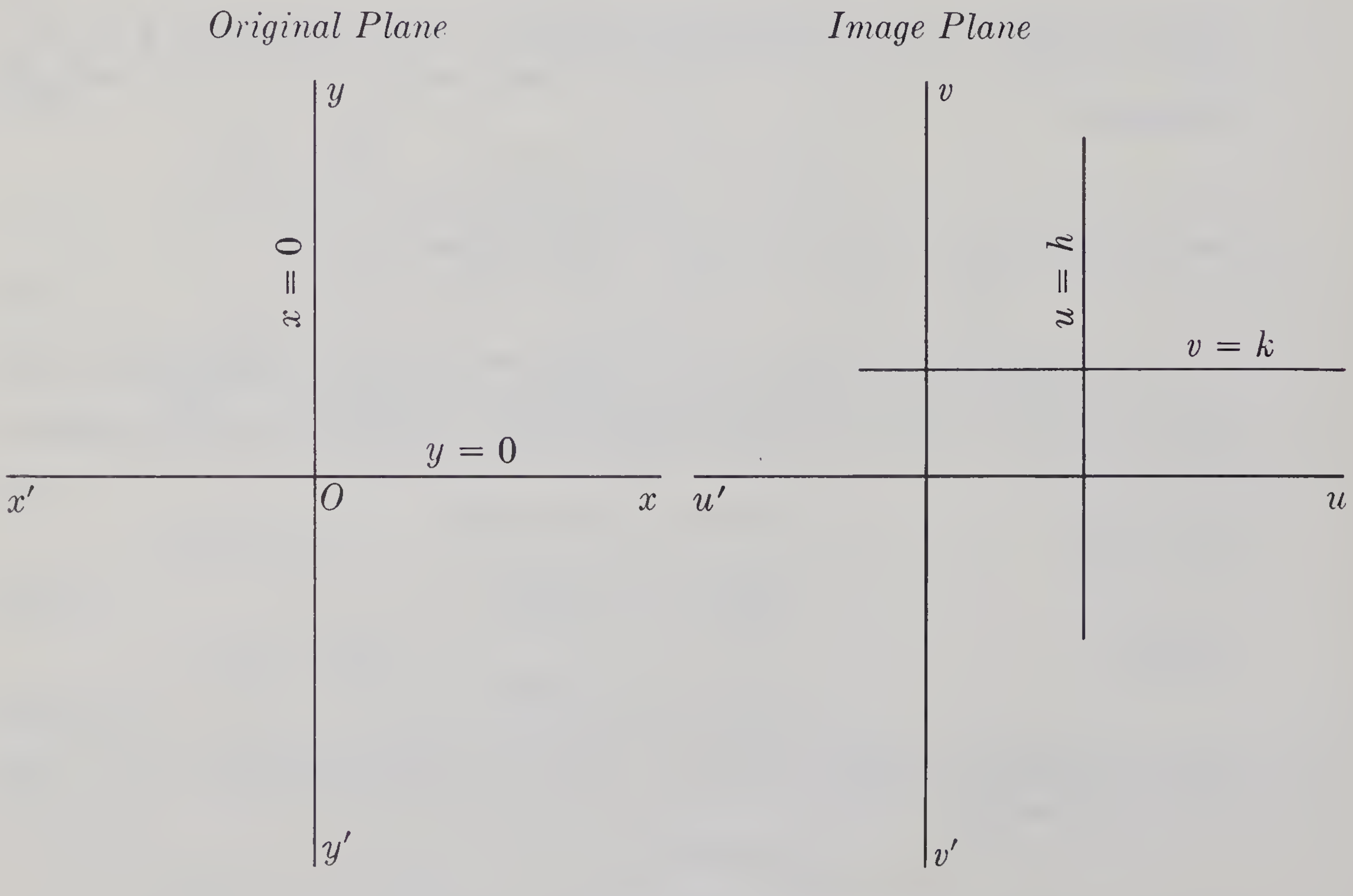


Figure 7.6

Note that the images of the axes in the original plane intersect at (h, k) in the image plane.

Let us now consider a line through the origin $(0, 0)$ in the original plane, for example, the line

$$y = 2x .$$

Example 2 in Section 7.1 indicated that points on a line are translated into points on a line with the same slope. We can verify this fact by plotting some points on $y = 2x$, for example $(0, 0)$, $(1, 2)$, $(2, 4)$, and their images in the image plane. Thus we see that the image of the line $y = 2x$ passes through the point (h, k), the image of the origin in the image plane, and has slope 2. The equation of such a line in the image plane where the co-ordinates are u and v is

$$v - k = 2(u - h) ,$$

as shown in Figure 7.7 for $(h, k) = (2, 1)$.

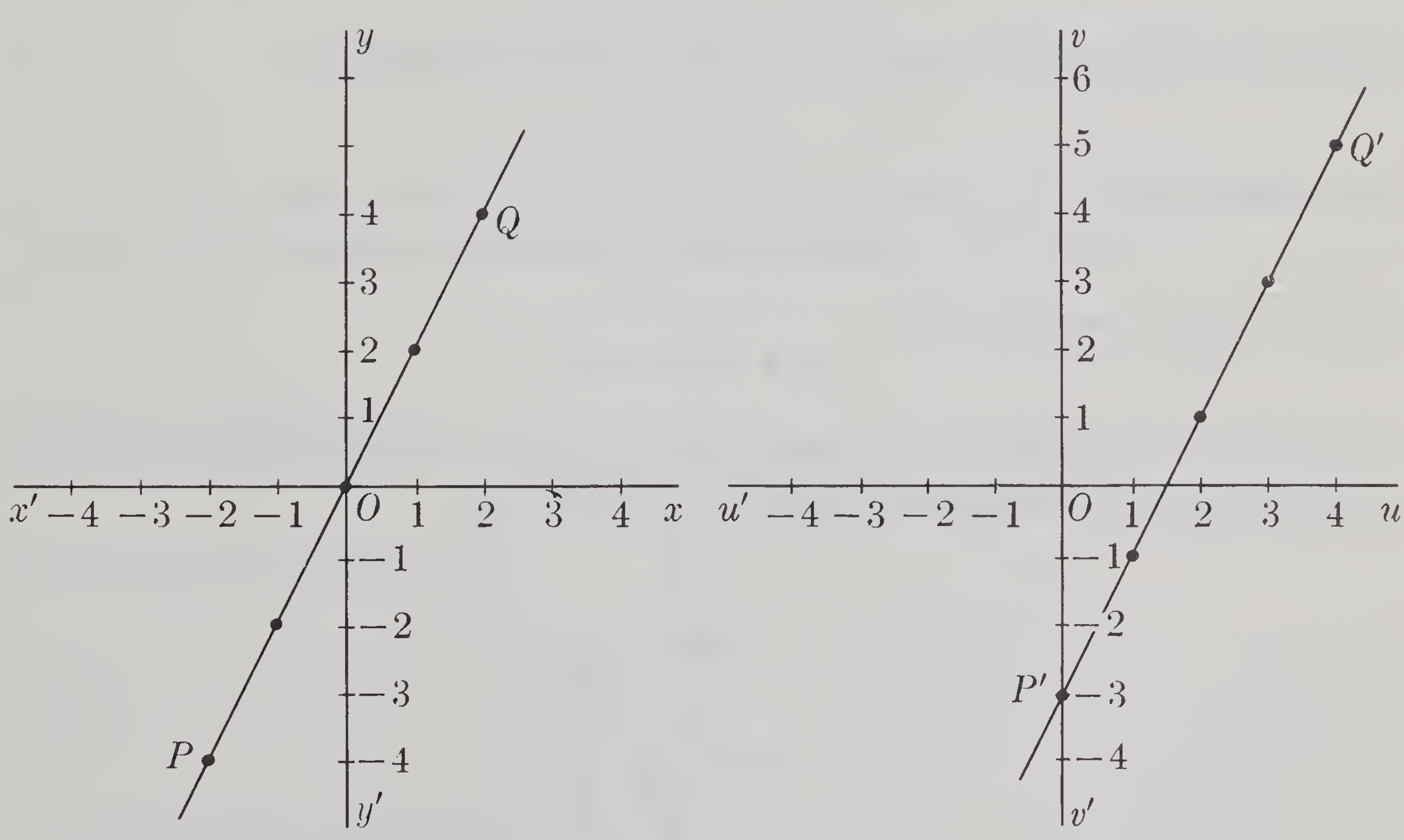


Figure 7.7

We have marked the image points P' and Q' corresponding to the points P and Q.

We notice immediately that we may obtain the equation of the image line

$$v - k = 2(u - h)$$

by using

$$u = x + h, \quad v = y + k,$$

to obtain

$$x = u - h, \quad y = v - k,$$

and then substituting these values in

$$y = 2x.$$

We may also represent this mapping on a single plane by superimposing the uv-axes (shown in red) on the xy-axes (shown in black), as we have done in Figure 7.8. In this case, $(h, k) = (2, 1)$, and the original line given by $y = 2x$ is translated into the image line given by $v - 1 = 2(u - 2)$.

As the uv-axes and xy-axes coincide in the single-plane representation the equation of the image line in the single plane may be written as

$$y - 1 = 2(x - 2).$$

Thus, in the single-plane representation the original line given by

$$y = 2x$$

is translated by

$$(x, y) \to (x + 2, y + 1)$$

into the image line given by

$$y - 1 = 2(x - 2).$$

The graph of the single-plane representation is shown in Figure 7.8, where again we have marked corresponding points P, Q and P', Q'.

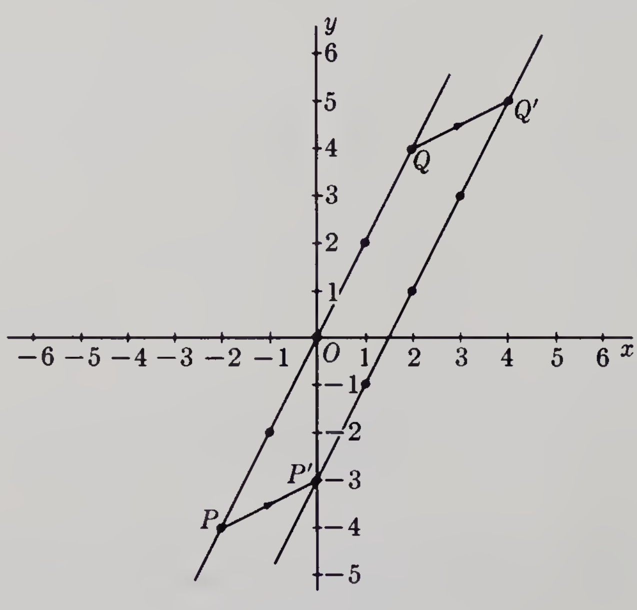


Figure 7.8

We shall now apply a similar analysis to a line that does not pass through the origin.

Example 1. Find the image of the line given by the equation

$$y = 3x - 2$$

under the translation

$$(x, y) \longrightarrow (x - 1, y + 1) = (u, v).$$

Solution:

$$u = x - 1, \quad v = y + 1;$$

therefore,

$$x = u + 1, \quad y = v - 1.$$

The image of

$$y = 3x - 2$$

is given by

$$v - 1 = 3(u + 1) - 2,$$
$$v - 1 = 3u + 3 - 2,$$
$$v = 3u + 2.$$

We may verify that this line is indeed the equation of the image line in the uv-plane, the image plane, by the following procedure. The line

$$y = 3x - 2$$

passes through the points

$$(0, -2) \text{ and } (1, 1).$$

The images of these points are

$$(0 - 1, -2 + 1) \text{ and } (1 - 1, 1 + 1),$$

that is,

$$(-1, -1) \text{ and } (0, 2).$$

We easily verify that the line

$$v = 3u + 2$$

passes through these two image points, and hence that

$$v = 3u + 2$$

is the equation of this image line.

Example 2. Find a translation

$$(x, y) \to (x + h, y + k)$$

that simplifies the equation of a line

$$y = 4x + 3 \quad \text{to} \quad y = 4x$$

in a single-plane representation.

Solution: Let the mapping be

$$(x, y) \to (x + h, y + k) = (u, v) ;$$

then

$$u = x + h , \quad v = y + k ;$$

therefore,

$$x = u - h , \quad y = v - k .$$

Substituting in $y = 4x + 3$, we obtain

$$v - k = 4(u - h) + 3 ,$$
$$v = 4u + k - 4h + 3 .$$

If

$$k - 4h + 3 = 0 ,$$

then the equation on the uv-plane becomes

$$v = 4u ,$$

Suitable values for h and k are

$$h = 0 , \quad k = -3 ,$$
$$h = \tfrac{3}{4} , \quad k = 0 ,$$
$$h = 1 , \quad k = 1 ,$$

and

$$h = 2 , \quad k = 5 ,$$

amongst others. Thus,

$$(x, y) \to (x + 1, y + 1)$$

is a suitable translation to simplify the equation

$$y = 4x + 3$$

to

$$y = 4x$$

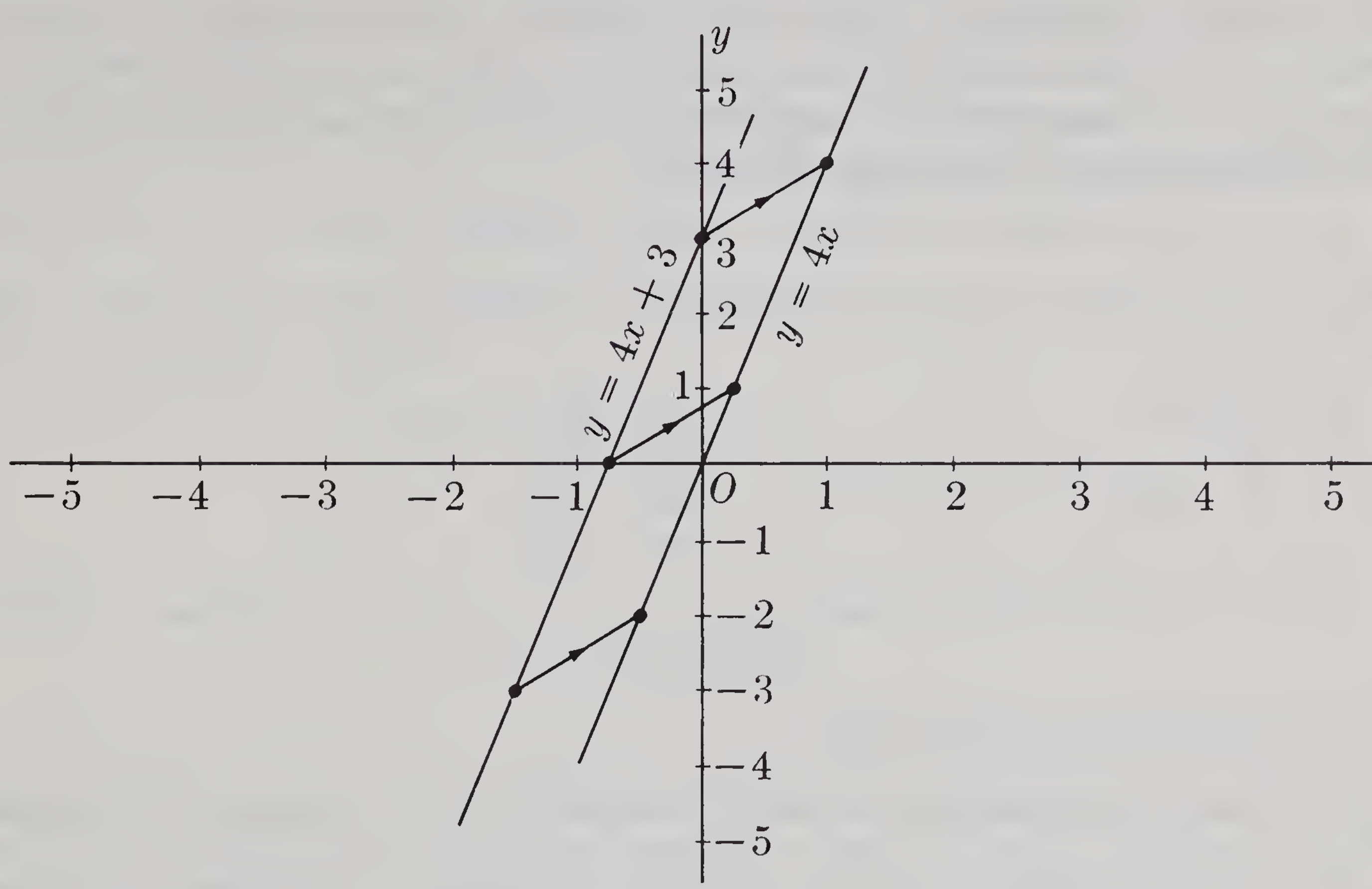
in the single-plane representation.

Other suitable translations are

$$(x, y) \to (x, y - 3)\ ,$$
$$(x, y) \to (x + \tfrac{3}{4}, y)\ ,$$

and

$$(x, y) \to (x + 2, y + 5)\ .$$



We show the single-plane representation for the mapping of

$$\{x, y) \mid y = 4x + 3\} \to \{(x, y) \mid y = 4x\}$$

by the translation

$$(x, y) \to (x + 1, y + 1)\ .$$

EXERCISE 7.3

Find the image of the line determined by each of the following equations under the translation stated, and sketch the graph in both planes.

1. $y = 3x\ ;\quad (x, y) \to (x - 2, y + 1) = (u, v)$
2. $y = -2x\ ;\quad (x, y) \to (x + 1, y + 3) = (u, v)$
3. $y = 4x + 1\ ;\quad (x, y) \to (x + 1, y + 3) = (u, v)$
4. $y = 2x - 3\ ;\quad (x, y) \to (x - 1, y + 1) = (u, v)$
5. $y = 2 - x\ ;\quad (x, y) \to (x - 2, y) = (u, v)$
6. $2y = 3x - 5\ ;\quad (x, y) \to (x - 1, y + 1) = (u, v)$

7. $2y = 3x - 5$; $(x, y) \to (x + 1, y + 4) = (u, v)$

8. $y = 2 - x$; $(x, y) \to (x, y + 2) = (u, v)$

9. $y = 2 - x$; $(x, y) \to (x - 1, y + 1) = (u, v)$

10. $y = 2x - \pi$; $(x, y) \to (x, y + \pi) = (u, v)$

Find suitable translations $(x, y) \to (x + h, y + k)$ that simplify the following equations for lines to the form $y = cx$ in a single-plane representation; give *at least two* such translations in each case. Show the positions of the original and image lines in a single-plane representation.

11. $y = 2x + 3$

12. $y = 4x - 1$

13. $2y = 5x - 7$

14. $3y = 1 - 4x$

15. $2x + 3y = 5$

16. $2x - 3y = 4$

17. $\frac{x}{2} + \frac{y}{3} = 1$

18. $\frac{y}{\sqrt{2}} = \frac{x}{\pi} + \frac{3}{\pi\sqrt{2}}$

19. Can the slope of a line, given by the value of m in the equation
$$y = mx + b,$$
be changed by a translation?

20. Can the intercepts of a line on the co-ordinate axes be changed by a translation?

21. (a) Find the image in the *uv*-plane of the region consisting of the points below the line
$$x + 2y = 5$$
under the translation
$$x \to x + 2, \quad y \to y - 1.$$
Is this region the same as the region consisting of the points below the image of the line?

(b) Make a similar comparison for points to the left of the line
$$3x + 2y = 7$$
under the translation
$$x \to x + 1, \quad y \to y + 3.$$

22. Generalize the results of the previous question for an arbitrary line under an arbitrary translation.

23. Find the images of the given regions under the given translations. Sketch the original region and its image.

(a) $x > 0, y < 0$; $x \to x + 1, y \to y - 1$

(b) $3y - 2x \leq 6, 2x - y < 2$; $x \to x - 3, y \to y + 7$

(c) $y - x < 1, x + 2y > 2, 3x + 2y < 6$; $x \to x - 5, y \to y - 3$

7.4. Geometric Invariance of a Locus Under a Translation

In Section 7.2, we saw that the length of a line segment and the angle between two line segments were invariant under a translation. In the last section, we found that the image of a line under a translation is a line. Each of these statements is equivalent to the statement that, in the cases considered, the image of the geometric figure under a translation is a congruent geometric figure. In fact, this statement is true for any geometric figure under a translation.

If the original geometric figure is a polygon, we can prove the congruency of its image under a translation by dividing the polygon into triangles and then proving that each triangle transforms into a congruent triangle.

Proof: Let ABC be any triangle in the original plane and $A'B'C'$ be its image under a translation. Then,

$$\left.\begin{aligned} AB &= A'B' , \\ BC &= B'C' , \\ CA &= C'A' . \end{aligned}\right\} \text{invariance of length}$$

Therefore,

$$\triangle ABC \equiv \triangle A'B'C' .$$

A rigorous proof for any curve is more difficult. However, we can fix the relative positions of *any* three points on a curve by a triangle and we see that the three corresponding points after the translation have the same relative positions. This makes reasonable the conclusion that the geometric character of any figure is invariant under a translation. Thus a circle translates into a circle of the same radius; an ellipse translates into an ellipse of the same dimensions; and a square translates into a square of the same size.

Example 1. Find the image of the parallelogram $ABCD$, for the points $A(0,0)$, $B(2,0)$, $C(1,3)$, $D(3,3)$ under the translation

$$(x, y) \to (x - 2 , y - 2) .$$

Show that the image is a parallelogram.

Solution:

$$\begin{aligned} &A(0,0) \to A'(-2 , -2) \\ &B(2,0) \to B'(0, -2) \\ &C(1,3) \to C'(-1,1) \\ &D(3,3) \to D'(1,1) \end{aligned}$$

$$\text{Slope of } A'B' = \frac{-2 - (-2)}{0 - (-2)} = 0 .$$

$$\text{Slope of } C'D' = \frac{1 - 1}{1 - (-1)} = 0 .$$

Therefore,

$$A'B' \parallel C'D' .$$

Similarly,

$$A'D' \parallel B'C' .$$

Hence, $A'B'C'D'$ is a parallelogram.

Example 2. Find the equation of the image of the circle

$$x^2 + y^2 = 9$$

under the translation

$$(x, y) \to (x + 1, y - 2) .$$

Solution: The given circle has radius 3 units and centre at $(0, 0)$ in the xy-plane. The point $(0, 0)$ in the xy-plane has the image $(1, -2)$ in the uv-plane under the given translation. The radius is invariant under the translation. Therefore, the image is a circle with centre at $(1, -2)$ and radius 3 units.

Let (u, v) be any point on this circle; then

$$\text{length of the radius} = \sqrt{(u - 1)^2 + (v - (-2))^2} .$$

Therefore,

$$(u - 1)^2 + (v + 2)^2 = 3^2 ,$$

$$u^2 - 2u + 1 + v^2 + 4v + 4 = 9 ,$$

or

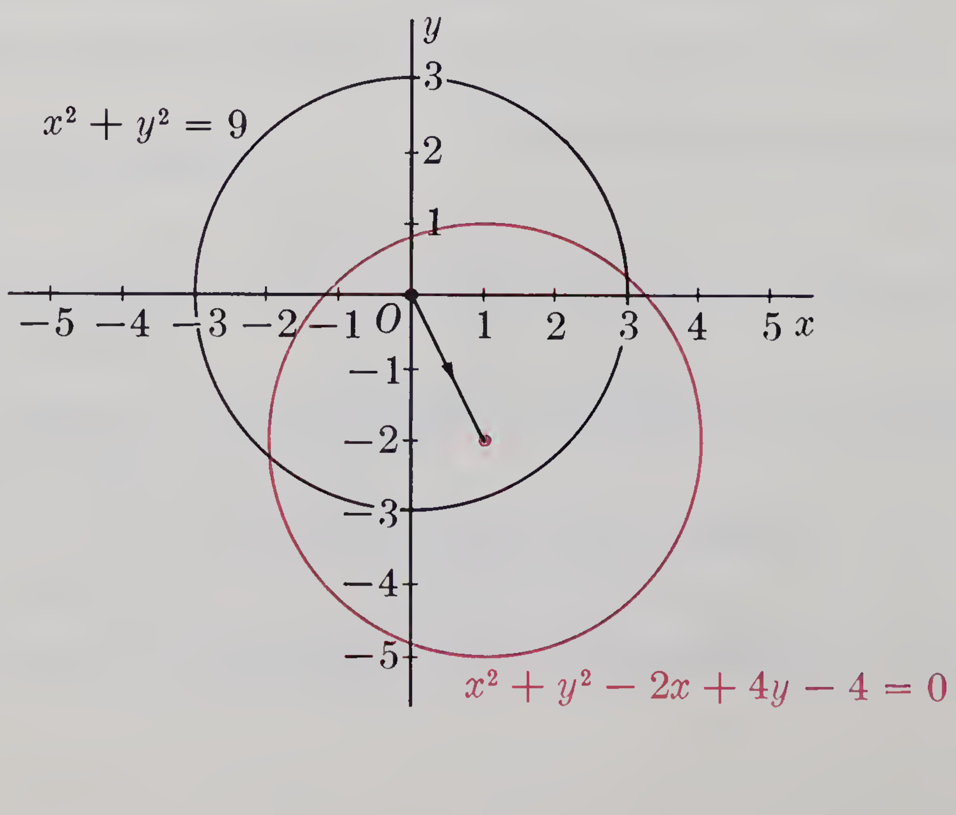
$$u^2 + v^2 - 2u + 4v - 4 = 0$$

is the equation of the image circle in the uv-plane.

In a single-plane representation, the image circle has the equation

$$x^2 + y^2 - 2x + 4y - 4 = 0 .$$

This mapping of the circle is shown in the diagram.



Alternative Solution:

$$(x, y) \to (x + 1, y - 2) = (u, v) .$$

Therefore,

$$x + 1 = u, \quad y - 2 = v ,$$

and

$$x = u - 1 , \quad y = v + 2 .$$

Substituting in the equation of the given circle, we have

$$(u - 1)^2 + (v + 2)^2 = 9 ,$$

$$u^2 - 2u + 1 + v^2 + 4v + 4 = 9 .$$

Therefore,

$$u^2 + v^2 - 2u + 4v - 4 = 0$$

is the equation of the image circle in the uv-plane.

EXERCISE 7.4

1. The points $A(0, 0)$, $B(3, 0)$, $C(3, 3)$, and $D(0, 3)$ determine a square. What are the co-ordinates of the images A', B', C', and D' under the translation

$$(x, y) \to (x - \tfrac{3}{2}, y - \tfrac{3}{2}) \ ?$$

   Verify that the image figure is also a square with sides 3 units.

2. Verify that the image of the circle whose equation is

$$x^2 + y^2 = 25$$

   under the translation

$$(x, y) \to (x + 2, y - 1)$$

   is a congruent circle.

3. Use a translation $(x, y) \to (u, v)$ to show that

$$x^2 + y^2 - 6x + 8y = 0$$

   is the equation of a circle of radius 5 units. Where is the centre of the circle in the xy-plane? Where is the centre in the uv-plane? Which translation (or translations) reduces the equation to the form

$$u^2 + v^2 = 25 \ ?$$

For each of the following equations of circles, find a translation that maps the centre onto the origin, and find the equation of the image circle. Sketch the graphs of both circles and show the mapping for the centres.

4. $x^2 + y^2 - 6x + 4y + 12 = 0$

5. $x^2 + y^2 - 4x - 6y - 18 = 0$

6. $x^2 + y^2 + 6x - 8y - 24 = 0$

7. $x^2 + y^2 - 24x - 10y = 0$

8. $x^2 + y^2 + 4\sqrt{2}x + 2y = 0$

9. The lines with the equations
$$y = 2x + 1 \quad \text{and} \quad 2y = 3x + 3$$
intersect at the point $(1, 3)$. Find a translation that reduces them to
$$v = 2u \quad \text{and} \quad 2v = 3u$$
respectively in the uv-plane. Use this translation to show that the line determined by
$$4x + 2y = 10$$
passes through the point of intersection of the first two lines.

10. Show that the effect of the translation
$$(x, y) \to (x + 1, y - 2) = (u, v)$$
followed by the translation
$$(u, v) \to (u - 2, v + 3) = (s, t)$$
is equivalent to the single translation
$$(x, y) \to (x - 1, y + 1) = (s, t)$$
for
(a) the points
$$(2, 3), (-1, 4), (0, 0),$$
(b) the line given by
$$x + 2y = 1,$$
(c) the circle
$$x^2 + y^2 = 4.$$

7.5. The Images of the Conic Sections under a Translation

As we have noted, the geometric character of a curve is unchanged under a translation. Therefore,

an ellipse translates into an ellipse,

an hyperbola translates into an hyperbola,

and a parabola translates into a parabola.

Example 1. Find the image of the ellipse

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

under the translation

$$(x, y) \to (x + 3, y - 2) = (u, v) .$$

Solution:

$$(x, y) \to (x + 3, y - 2) = (u, v) ;$$

therefore,

$$x + 3 = u, \quad y - 2 = v ,$$

and

$$x = u - 3, \quad y = v + 2 .$$

Therefore,

$$\frac{(u - 3)^2}{9} + \frac{(v + 2)^2}{4} = 1 ,$$

$$4(u^2 - 6u + 9) + 9(v^2 + 4v + 4) = 36 ,$$

or

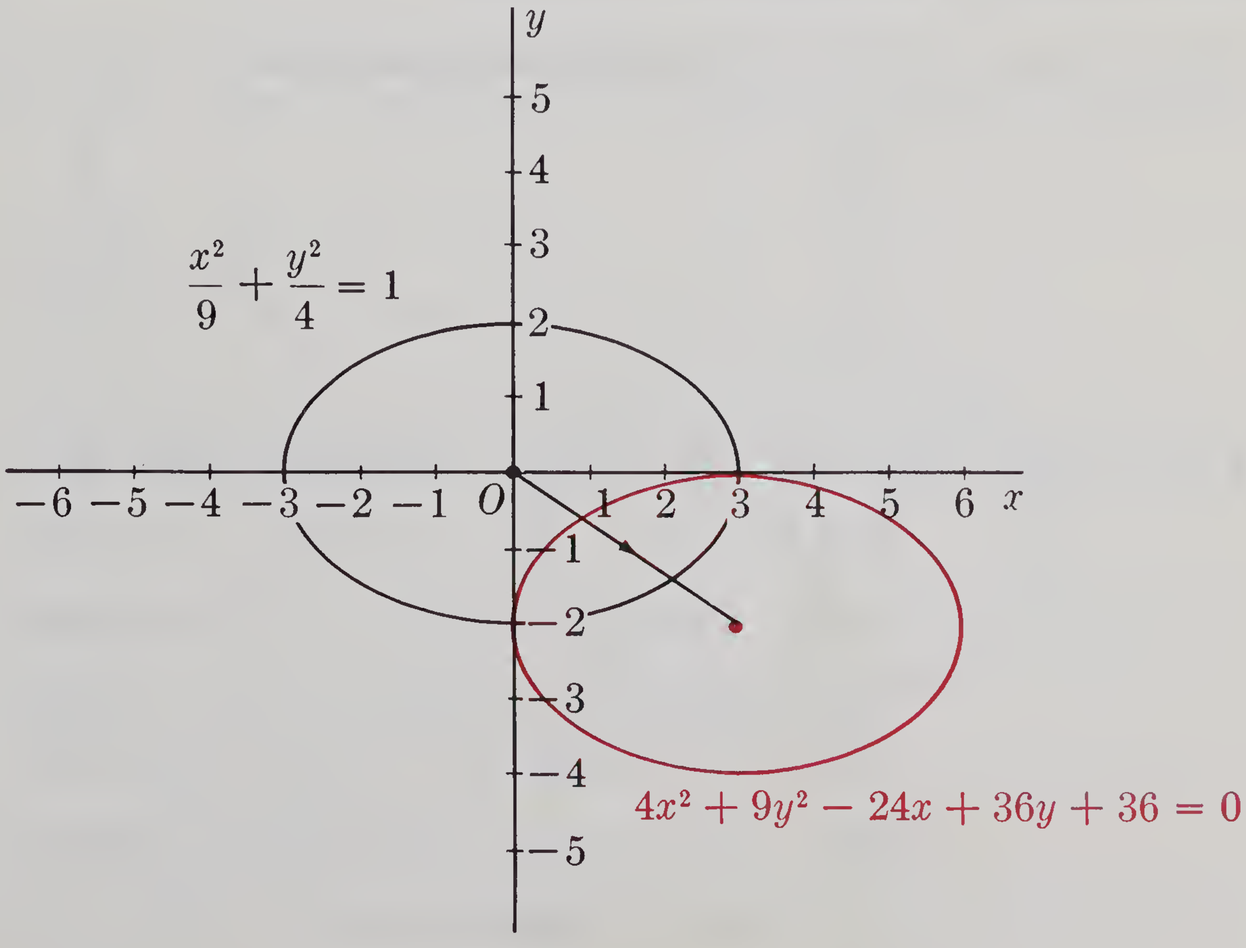
$$4u^2 + 9v^2 - 24u + 36v + 36 = 0$$

is the equation of the image ellipse in the uv-plane.

In a single-plane representation the image ellipse is

$$4x^2 + 9y^2 - 24x + 36y + 36 = 0 ,$$

as shown in the diagram.



Example 2. Use the transformation

$$(x, y) \to (x - 1, y + 1) = (u, v)$$

to show that

$$x^2 + 2y^2 - 2x + 4y - 5 = 0$$

is the equation of an ellipse.

Solution:

$$x^2 + 2y^2 - 2x + 4y - 5 = 0\,,$$
$$x^2 - 2x + 2y^2 + 4y = 5\,,$$
$$(x^2 - 2x + 1) + 2(y^2 + 2y + 1) = 8\,,$$
$$\frac{(x-1)^2}{8} + \frac{(y+1)^2}{4} = 1\,.$$

But

$$x - 1 = u \quad\text{and}\quad y + 1 = v\,;$$

therefore,

$$\frac{u^2}{8} + \frac{v^2}{4} = 1\,.$$

This equation is the standard equation of an ellipse of semiaxes $2\sqrt{2}$ and 2, with its centre at the origin in the uv-plane.

Translations leave geometric loci invariant; hence,

$$x^2 + 2y^2 - 2x + 4y - 5 = 0$$

is the equation of an ellipse in the xy-plane.

Two-Plane Representation

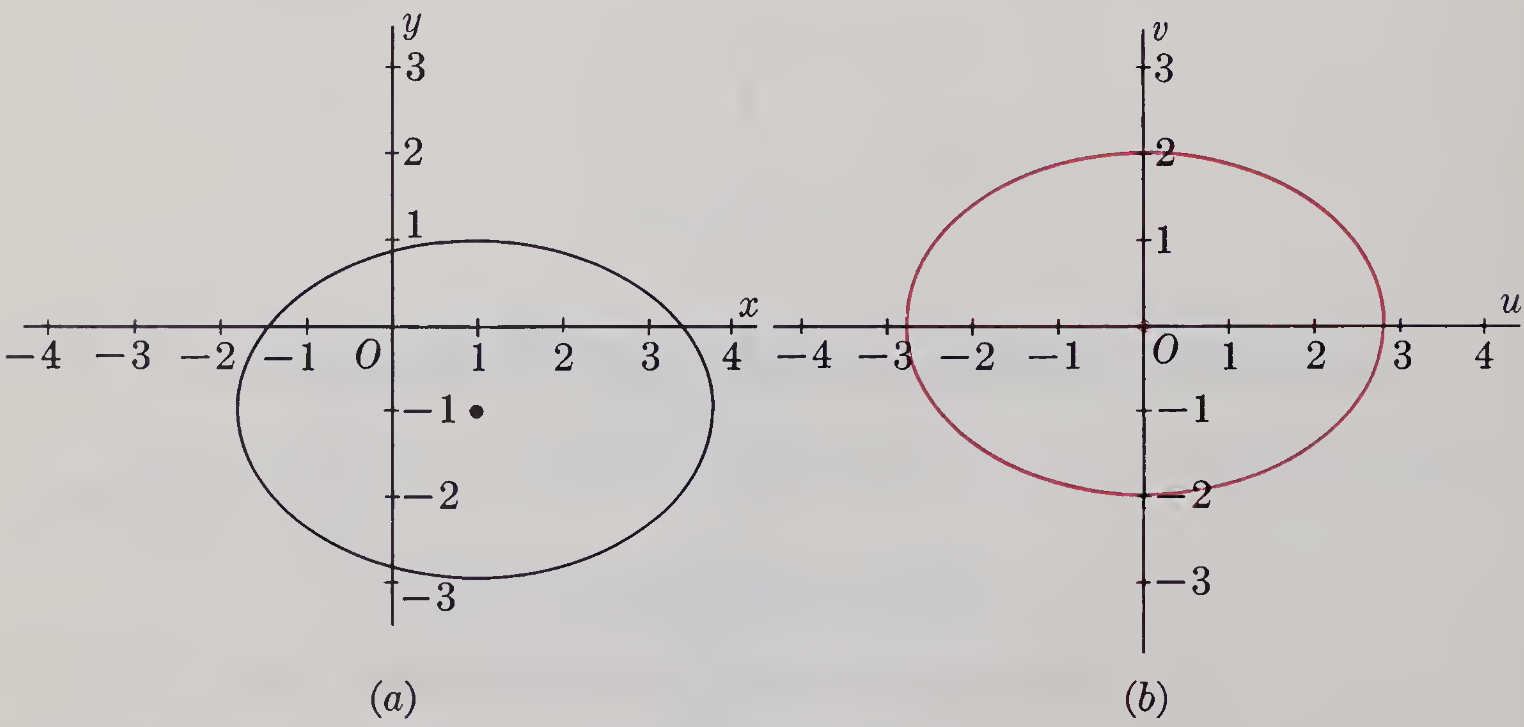
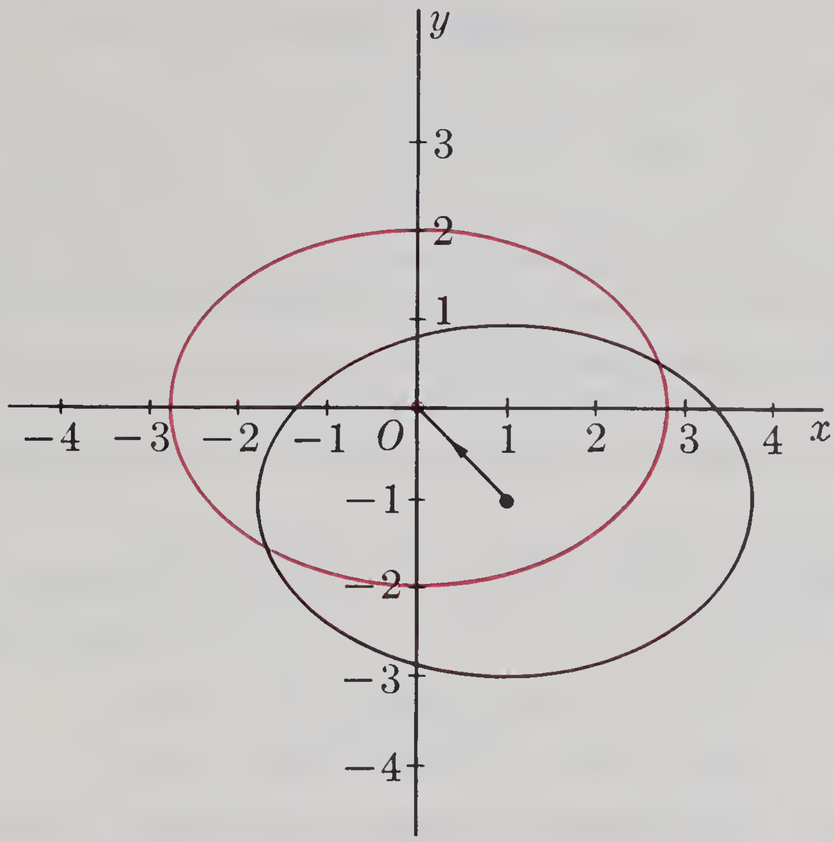


Figure 7.9

Note that the semiaxes of this ellipse must also be $2\sqrt{2}$ and 2, and the centre must be at the image of the origin of the uv-plane in the xy-plane, that is, at the point $(1, -1)$ in the xy-plane.

Single-Plane Representation



EXERCISE 7.5

Find the images of each of the curves determined by the following equations under the translations stated. State the type of curve determined by the equation in each case.

1. $y^2 - 4x + 8 = 0;\quad (x, y) \to (x - 2, y)$
2. $y^2 - 2x^2 - 2y + 6x = 12;\quad (x, y) \to (x - \frac{3}{2}, y - 1)$
3. $x^2 - 4y^2 + 8x + 24y = 20;\quad (x, y) \to (x + 4, y - 3)$
4. $2x^2 + 3y^2 - 12x + 12y + 24 = 0;\quad (x, y) \to (x - 3, y + 2)$
5. $9x^2 - 4y^2 + 54x - 8y - 41 = 0;\quad (x, y) \to (x + 3, y + 1)$
6. $3x^2 - y^2 + 6x + 4y - 1 = 0;\quad (x, y) \to (x + 1, y - 2)$
7. $4x^2 + 32x - 4y - 16 = 0;\quad (x, y) \to (x+4, y + 20)$
8. $x^2 + 4y^2 - 10x + 12y - 20 = 0;\quad (x, y) \to (x - 5, y + \frac{3}{2})$
9. $4x^2 + 9y^2 + 24x - 36y + 36 = 0;\quad (x, y) \to (x +3, y - 2)$
10. $9y^2 - 4x^2 - 18y + 8x - 8 = 0;\quad (x, y) \to (x - 1, y - 1)$

7.6. Simplification of Quadratic Relations by Translations

In the previous section, we have seen how a certain transformation may simplify a given quadratic relation. In example 2, the given translation

$$(x, y) \to (x - 1,\ y + 2) = (u, v)$$

simplified

$$\{(x, y) \mid x^2 + 2y^2 - 2x + 4y - 5 = 0\}$$

in the xy-plane to

$$\left\{(u, v) \,\middle|\, \frac{u^2}{8} + \frac{v^2}{4} = 1\right\},$$

in the uv-plane and hence we showed that the graph was an ellipse.
In this section, we illustrate how to find a translation that will perform the simplification. Basically, the method involves *completion of the square.*

Example 1. Find a translation that will eliminate the first degree terms from the equation

$$25x^2 - 16y^2 + 100x + 32y - 316 = 0\,.$$

Solution:

$$25x^2 - 16y^2 + 100x + 32y = 316\,,$$

$$25x^2 + 100x - (16y^2 - 32y) = 316\,,$$

$$25(x^2 + 4x) - 16(y^2 - 2y) = 316\,,$$

$$25(x^2 + 4x + 4) - 16(y^2 - 2y + 1) = 316 + 100 - 16\,,$$

$$25(x + 2)^2 - 16(y - 1)^2 = 400\,.$$

If

$$x + 2 = u, \qquad y - 1 = v\,,$$

then

$$25u^2 - 16v^2 = 400\,.$$

The required translation is

$$(x, y) \to (x + 2,\ y - 1) = (u, v)\,.$$

Note that the graph of the equation is a hyperbola with semi-axes 4 and 5; the equation in standard form is

$$\frac{u^2}{4^2} - \frac{v^2}{5^2} = 1\,.$$

This image under the translation

$$(x, y) \to (x + 2,\ y - 1)$$

is the hyperbola

$$\frac{x^2}{4^2} - \frac{y^2}{5^2} = 1$$

in the single-plane representation.

Example 2. Simplify

$$x^2 + 6x - 8y + 25 = 0$$

by a translation and identify the character of the curve. Show both the original curve and the image curve in a single-plane representation.

Solution:

$$(x^2 + 6x) - 8y + 25 = 0\,,$$
$$(x^2 + 6x + 9) = 8y - 25 + 9\,,$$
$$(x + 3)^2 = 8(y - 2)\,.$$

Put

$$u = x + 3\,, \quad v = y - 2\,;$$

then

$$u^2 = 8v$$

in the uv-plane. Therefore, in a single-plane representation, the image is given by

$$x^2 = 8y\,.$$

The image curve is a parabola with $x = 0$ for its axis, its vertex at $(x, y) = (0, 0)$, and opening towards positive values of y. Hence the original graph is a parabola with

$$x + 3 = 0$$

as its axis, its vertex at $(-3, 2)$ and opening towards positive values of y.

EXERCISE 7.6

For each of the following equations, find a translation that reduces the equation to the form

$$ax^2 + by^2 = c \qquad \text{or} \qquad y^2 = 4ax \qquad \text{where} \quad a, b, c \in I\,.$$

State the type of conic section represented in each case.

1. $y^2 + 4x - 4y + 8 = 0.$
2. $4x^2 + 32x - 4y - 13 = 0.$
3. $y = 4x - x^2.$
4. $x^2 - 6x - 4y + 8 = 0.$
5. $x^2 - y^2 + 12x + 4y + 16 = 0.$
6. $3x^2 + 4y^2 + 12x + 6y + 7 = 0.$
7. $4x^2 + y^2 + 12x - 10y + 14 = 0.$
8. $x^2 - y^2 + 8x - 14y - 35 = 0.$
9. $9x^2 + 4y^2 + 36x - 24y + 36 = 0.$
10. $4x^2 - 9y^2 - 8x + 72y - 104 = 0.$
11. By examining the pattern of the results in questions (1) to (10), can you form a rule for the type of curve that is given by an original equation of the form

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

when (i) $ab = 0$,
(ii) $ab > 0$,
(iii) $ab < 0$?

12. The mapping

$$(x, y) \to (x + h, y + k)$$

reduces

$$ax^2 + by^2 + 2gx + 2fy + c = 0$$

to a standard form for a conic. Find the values of h and k in terms of a, b, g, and f. Verify that

(a) if $ab = 0$ the conic is a parabola,

(b) if $ab > 0$ the conic is an ellipse,

(c) if $ab < 0$ the conic is a hyperbola.

Chapter Summary

The general translation $(x, y) \to (x + h, y + k) = (u, v)$ · Invariance of length and angle under translation · Invariance of locus shape under translation · Simplification of quadratic relations by translations

REVIEW EXERCISE 7

1. Find the images of the points

$$A\,(2, 6), \quad B\,(1, 1), \quad \text{and} \quad C\,(-3, -19)$$

under the translation

$$(x, y) \to (x + 2, y - 4)\,.$$

Show that (a) A, B, C are collinear, (b) the images A', B', C' are collinear, and (c) the slope of the line is invariant under the translation.

2. What geometric figure is formed by the points

$$A\,(3, 4), \quad B\,(0, 0), \quad \text{and} \quad C\,(6, 0)\ ?$$

Find the images of A, B, and C under the translation

$$(x, y) \to (x - 4, y - 2)$$

and prove that ABC and $A'B'C'$ are congruent.

3. Find the images of the points

$$A\,(2, \sqrt{3}), \quad B\,(1, 1) \quad \text{and} \quad C\,(5, -\sqrt{3})$$

under the translation

$$(x, y) \to (x - 2, y - \sqrt{3})\,.$$

Show that angle ABC = angle $A'B'C'$.

4. (a) Show that the image of the line whose equation is

$$y = 5x - 4$$

under the translation

$$(x, y) \to (x - 3, y + 4) = (u, v)$$

is the line whose equation is

$$v = 5u + 15 .$$

(b) Verify that

$$v = 5u + 15$$

is the equation of the image line by finding two points that lie on the original line and then finding the equation of the line that passes through the images of these two points.

5. Find the equation of the image of the line determined by

$$3y = 2x - 7$$

under the translation

$$(x, y) \to (x + 4, y - 5) = (u, v) .$$

Find suitable translations

$$(x, y) \to (x + h, y + k)$$

that simplify the following equations for lines to the form

$$y = kx .$$

Find at least one such translation in each case and show the original and the image lines in a single-plane representation.

6. $y = 4x - 3$

7. $2x + y = 5$

8. $\frac{x}{3} + \frac{y}{4} = 1$

9. $\sqrt{3}y = \sqrt{2}x + \sqrt{3}$

10. Find the image in the uv-plane of the region determined by

$$3x - 5y < 15$$

under the translation

$$(x, y) \to (x - 2, y - 1) .$$

Sketch the region in the xy- and uv-planes.

11. (a) Sketch the region consisting of the points under the line whose equation is

$$x - 3y + 3 = 0 .$$

(b) Find the equation in the uv-plane of the image of the line in (a) under the translation

$$(x, y) \to (x - 1, y + 2) .$$

Is the region consisting of the points under the line determined in (b) the same region as the region which is the image of the region in (a)?

12. Verify that the image of the circle whose equation is
$$4x^2 + 4y^2 = 25$$
under the translation
$$(x, y) \rightarrow (x + 3, y - 2)$$
is a congruent circle.

13. For the circle determined by
$$x^2 + y^2 - 10x + 6y + 22 = 0 .$$
find a translation that maps the centre onto the origin, and find the equation of the image circle. Show the graphs of both circles in a single-plane representation.

14. Repeat question (13) for
$$4x^2 + 4y^2 - 4x + 12y = 15 .$$

15. Find the image of the conic determined by
$$4x^2 - 9y^2 + 32x + 18y + 19 = 0$$
under the translation
$$(x, y) \rightarrow (x + 4, y - 1) .$$
Name and describe the conic.

16. Repeat question (15) for
$$x^2 + 4y^2 + 8x - 6y + 12 = 0$$
and the translation
$$(x, y) \rightarrow (x + 4, y - \tfrac{3}{4}) .$$

For each of the following equations, find a translation that reduces the equations to the form
$$ax^2 + by^2 = c \qquad \text{or} \qquad y^2 = 4ax$$
where a, b, $c \in I$. Identify the conic section represented in each case.

17. $y^2 + 4x - 8y + 28 = 0$

18. $3x^2 + 4y^2 - 30x - 8y + 67 = 0$

19. $3x^2 + y^2 - 18x + 10y - 10 = 0$

20. $16x^2 + 7y^2 - 32x + 28y - 68 = 0$

21. $4x^2 - 12x - 16y + 1 = 0$

22. Show that the effect of the translation
$$(x, y) \rightarrow (x - 4, y - 2) = (u, v)$$
followed by the translation
$$(u, v) \rightarrow (u + 5, v + 1) = (s, t)$$
is equivalent to the single translation
$$(x, y) \rightarrow (x + 1, y - 1) = (s, t)$$
for (a) the line $5x - y = 4$, (b) the circle $4x^2 + 4y^2 = 49$, and (c) the hyperbola $2x^2 - 9y^2 = 18$.

Chapter 8

ROTATIONS IN THE PLANE

$x = r\cos\alpha$ $\quad x' = r\cos(\alpha+\theta)$

$y = r\sin\alpha$ $\quad = r\cos\alpha\cos\theta - r\sin\alpha\sin\theta$

$= x\cos\theta - y\sin\theta$

8.1. The Mapping $(x, y) \to (x \cos \theta - y \sin \theta,\ \ x \sin \theta + y \cos \theta)$

If we consider a two plane representation of the transformation

$$(x, y) \to \left(x \cos \frac{\pi}{4} - y \sin \frac{\pi}{4},\ x \sin \frac{\pi}{4} + y \cos \frac{\pi}{4}\right) = (u, v)\,,$$

that is

$$(x, y) \to \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}},\ \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right) = (u, v)\,,$$

we see that

$$(0, 0) \to (0, 0)\,,$$

$$(1, 0) \to \left(\frac{1}{\sqrt{2}}\ \ \frac{1}{\sqrt{2}}\right),$$

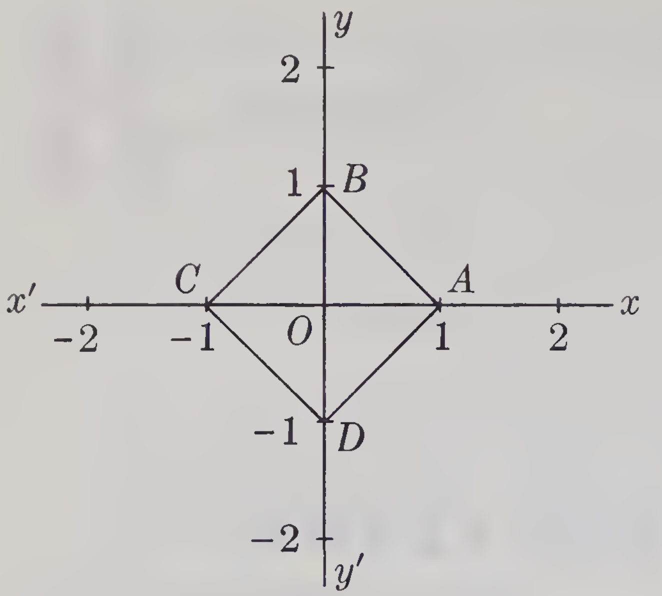
$$(0, 1) \to \left(-\frac{1}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right),$$

$$(-1, 0) \to \left(-\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right),$$

and

$$(0, -1) \to \left(\frac{1}{\sqrt{2}},\ -\frac{1}{\sqrt{2}}\right).$$

The two-plane representation of the transformation of these points is shown in Figure 8.1.



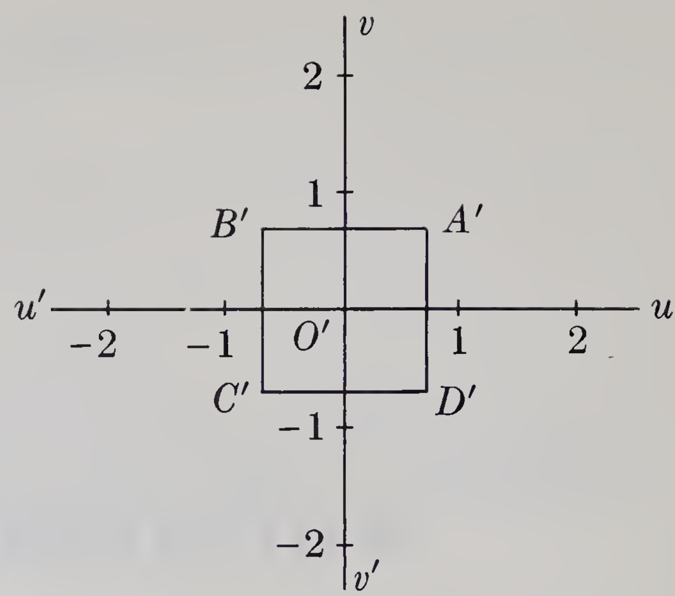


Figure 8.1

We see immediately that each of the original points O, A, B, C, D in the xy-plane is mapped onto a unique image point O', A', B', C', D' in the uv-plane. Obviously, *any* given point (a, b) in the xy-plane is mapped onto a unique image point $\left(\frac{a}{\sqrt{2}} + \frac{b}{\sqrt{2}}, \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}}\right)$ in the uv-plane.

Thus, the transformation

$$(x, y) \to \left(\frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}, \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}\right)$$

maps the points in the xy-plane onto corresponding points in the uv-plane and the correspondence is one-to-one. (See Exercise 8.1, question (7).)

As in the case of the translations, we can use a one-plane representation if we make the xy-axes and uv-axes coincide. Figure 8.2 shows the same points as Figure 8.1 under this transformation.

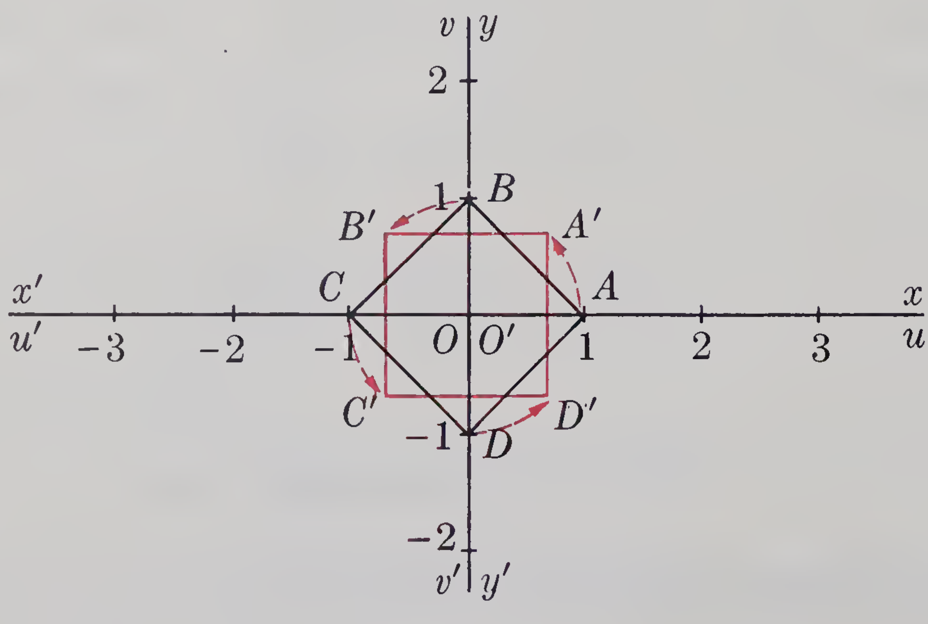


Figure 8.2

This last diagram seems to indicate that the effect of the transformation on the square $ABCD$ is the *rotation* of the square about the origin O through an angle of $\frac{\pi}{4}$ into the congruent square $A'B'C'D'$. (Verify that $ABCD$ and $A'B'C'D'$ are congruent squares.) This rotation does indeed occur and the effect of the transformation is the rotation of the original plane through an angle of $\frac{\pi}{4}$ about the origin, which is a fixed point for the transformation and, in fact, is the only fixed point.

The general transformation of this kind is

$$P \to P',$$
$$(x, y) \to (x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta) = (u, v)\,.$$

We see that, for this transformation,

$$\begin{aligned} OP^2 &= (x-0)^2 + (y-0)^2 \\ &= x^2 + y^2\,, \end{aligned}$$

and

$$\begin{aligned} OP'^2 &= (u-0)^2 + (v-0)^2 \\ &= (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 \\ &= x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta + x^2\sin^2\theta \\ &\quad + 2xy\sin\theta\cos\theta + y^2\cos^2\theta \\ &= x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos^2\theta) \\ &= x^2 + y^2 \\ &= OP^2\,. \end{aligned}$$

Thus, the transformation is such that the distance of any point from the origin is constant under the transformation. Also, if we consider two points $P(a, b)$ and $Q(c, d)$ and their image points $P'(a', b')$ and $Q'(c', d')$, then

$$a' = a\cos\theta - b\sin\theta,\quad b' = a\sin\theta + b\cos\theta$$

and

$$c' = c\cos\theta - d\sin\theta,\quad d' = c\sin\theta + d\cos\theta\,.$$

Therefore,

$$PQ^2 = (a-c)^2 + (b-d)^2\,,$$

and

$$\begin{aligned} P'Q'^2 &= (a'-c')^2 + (b'-d')^2 \\ &= [(a-c)\cos\theta - (b-d)\sin\theta]^2 + [(a-c)\sin\theta + (b-d)\cos\theta]^2 \\ &= (a-c)^2(\cos^2\theta + \sin^2\theta) + (b-d)^2(\sin^2\theta + \cos^2\theta) \\ &= PQ^2\,. \end{aligned}$$

Thus, the lengths of line segments are invariant under the transformation

These two results together show that any point on a circle with its centre at the origin is mapped into another point on the same circle and that the distance the point is "moved" on the circle is the same for each point on the circle. Such a "motion" is produced by a rigid counterclockwise rotation of the plane through an angle θ about the origin as the centre of rotation. For this reason, the transformation is called a rotation.

DEFINITION. The mapping

$$(x, y) \to (x \cos \theta - y \sin \theta,\ x \sin \theta + y \cos \theta), \qquad x, y, \theta \in Re$$

is a one-to-one mapping called a *rotation*. The rotation is counterclockwise through the angle θ radians; the centre of rotation is $(0, 0)$.

Example. Find the image points of some typical points on the line $y = x$ under the transformation

$$(x, y) \to (\tfrac{\sqrt{3}}{2}x - \tfrac{1}{2}y,\ \tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y)\,.$$

Using a single-plane representation, illustrate this mapping as a rotation of the line $y = x$ through an angle of 30°. Indicate also the image of the half plane $x > y$.

Solution: Under the transformation

$$(x, y) \to (\tfrac{\sqrt{3}}{2}x - \tfrac{1}{2}y,\ \tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y)$$

we obtain the following correspondences:

$$(0, 0) \to (0, 0)\,,$$

$$(1, 1) \to \left(\frac{\sqrt{3} - 1}{2},\ \frac{1 + \sqrt{3}}{2}\right),$$

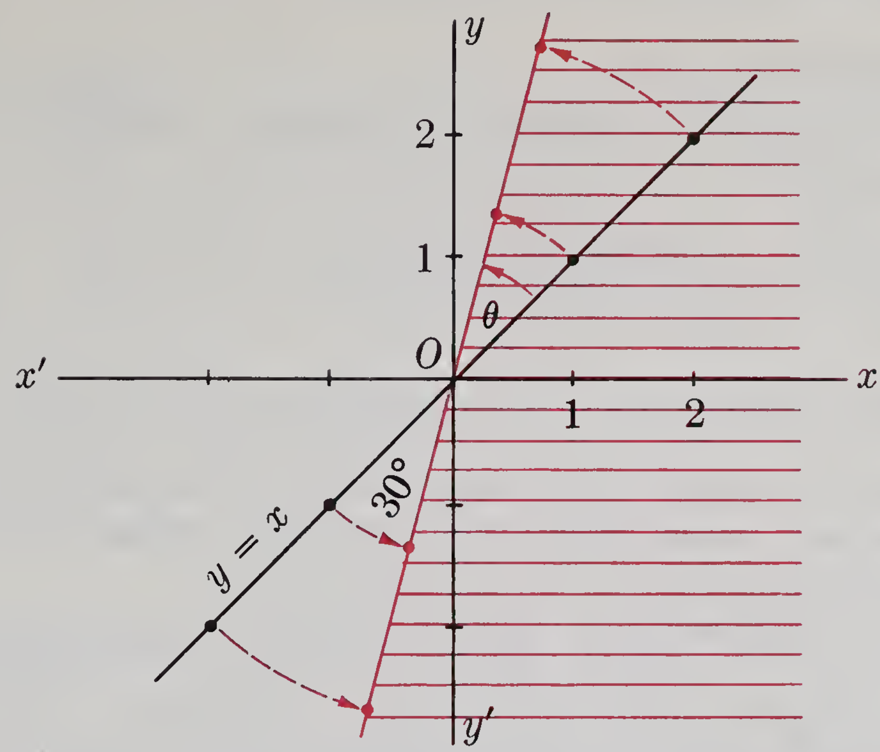
$$(2, 2) \to (\sqrt{3} - 1,\ 1 + \sqrt{3})\,,$$

$$(-1, -1) \to \left(\frac{-\sqrt{3} + 1}{2},\ \frac{-\sqrt{3} - 1}{2}\right),$$

$$(-2, -2) \to (-\sqrt{3} + 1,\ -\sqrt{3} - 1)\,.$$

Thus, the image points lie on a line through the origin with slope $\dfrac{1 + \sqrt{3}}{\sqrt{3} - 1}$. The equation of the image line is

$$(\sqrt{3} - 1)y = (1 + \sqrt{3})x\,.$$



The Image of the Half Plane $x > y$

The slope of the original line, $y = x$, is 1, the slope of the image line is $\frac{1+\sqrt{3}}{\sqrt{3}-1}$.
Therefore, if ϕ is the angle between the lines, we have

$$\tan\phi = \frac{\dfrac{1+\sqrt{3}}{\sqrt{3}-1} - 1}{1 + \dfrac{1+\sqrt{3}}{\sqrt{3}-1}}$$

$$= \frac{\sqrt{3}+1-\sqrt{3}+1}{\sqrt{3}-1+1+\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}},$$

$$\phi = \frac{\pi}{6} \text{ or } 30^\circ .$$

EXERCISE 8.1

1. Find the images of the points
$$A(4, 3), \quad B(-1, 3), \quad C(-1, -2), \quad D(4, -2),$$
under the rotations

(a) $(x, y) \to \left(\dfrac{x-y}{\sqrt{2}}, \dfrac{x+y}{\sqrt{2}}\right)$, (b) $(x, y) \to \left(\dfrac{-x-y}{\sqrt{2}}, \dfrac{x-y}{\sqrt{2}}\right)$,

(c) $(x, y) \to \left(\dfrac{x}{2} - \dfrac{\sqrt{3}y}{2}, \dfrac{\sqrt{3}x}{2} + \dfrac{y}{2}\right)$, (d) $(x, y) \to (-y, x)$.

State the angle of rotation in each case. Show the corresponding points in both two-plane and single-plane representations.

2. Find the images of the points

$$A(-2, -1), \quad B(0,0), \quad C(2,1), \quad D(6,3),$$

under the rotations

(a) $(x, y) \to \left(\dfrac{\sqrt{3}x - y}{2}, \dfrac{x + \sqrt{3}y}{2}\right)$, (b) $(x, y) \to \left(\dfrac{x+y}{\sqrt{2}}, \dfrac{-x+y}{\sqrt{2}}\right)$,

(c) $(x, y) \to (-x, -y)$, (d) $(x, y) \to \left(\dfrac{-x - \sqrt{3}y}{2}, \dfrac{\sqrt{3}x - y}{2}\right)$.

State the angle of rotation in each case and show the corresponding points in both two-plane and single-plane representations. What do you note about the set of given points and the set of image points?

3. The points A (2,3), $B(-1, +1)$, $C(0, -3)$, and $D(3, -1)$ determine a parallelogram $ABCD$. Find the length of the sides AB and AD. The parallelogram is transformed into another quadrilateral $A'B'C'D'$ by the rotation

$$(x, y) \to \left(\frac{x\sqrt{3} - y}{2}, \frac{x + y\sqrt{3}}{2}\right).$$

Show that $A'B'C'D'$ is also a parallelogram congruent to $ABCD$. Sketch the two parallelograms in a single plane.

4. The points $A\left(0, -\frac{\sqrt{3}}{3}\right)$, $B\left(-\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$, and $C\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$ determine an equilateral triangle with its centroid (point of intersection of the medians) at (0, 0). Find two *different* rotations which transform $\triangle ABC$ into a congruent triangle coincident with $\triangle ABC$. State the mapping for each transformation in the standard form.

5. If $P(x, y)$ and $P'(u, v)$ are points in the plane such that

$$OP = OP',$$

and

$$\angle POP' = \theta,$$

show that

$$u = x\cos\theta - y\sin\theta$$

and

$$v = x\sin\theta + y\cos\theta.$$

(*Hint:* Let angle xOP *be* ψ and angle xOP' be ϕ and apply the formulae for the sine and cosine of the sum of two angles.)

6. If $P(x, y)$ and $P'(u, v)$ are points in the plane such that

$$u = x\cos\theta - y\sin\theta$$

and

$$v = x\sin\theta + y\cos\theta,$$

show that

$$\angle POP' = \theta.$$

(*Hint:* Find $\tan \angle POP'$.)

7. Given the mapping
$f\colon (x, y) \to (x \cos \theta - y \sin \theta,\ x \sin \theta + y \cos \theta) = (u, v), \qquad (x, y, \theta \in \mathrm{Re})$
where (u, v) is the image of (x, y), show that
$$x = u \cos \theta + v \sin \theta\,,$$
$$y = -u \sin \theta + v \cos \theta\,.$$
Now let
$$g\colon (u, v) \to (u \cos \theta + v \sin \theta,\ -u \sin \theta + v \cos \theta)$$
and show that $g(f(x, y)) = (x, y)$.

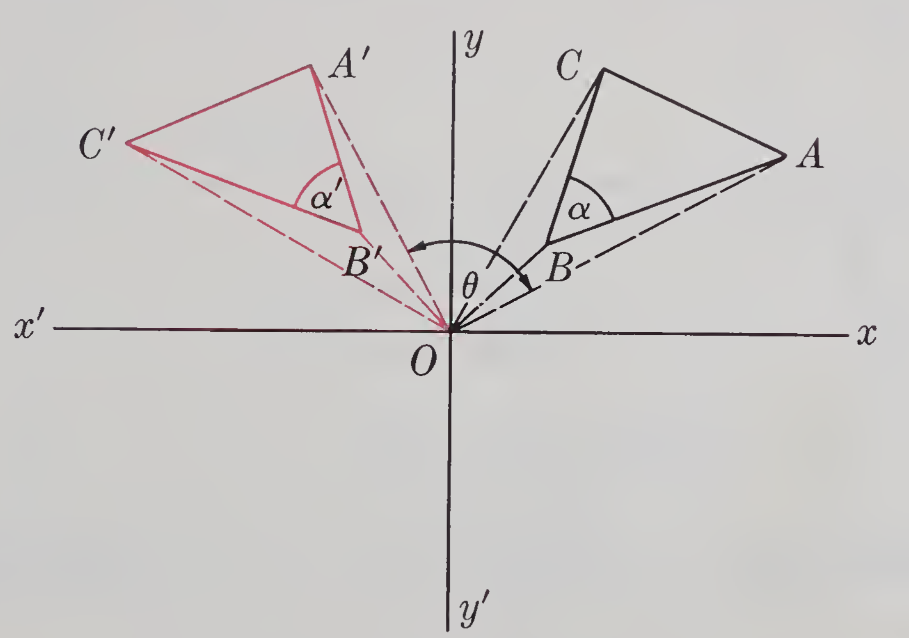
8.2. Geometric Invariance Under a Rotation

We have already shown that the distance between two points P and Q is unchanged or *invariant* under a rotation. If we also show that the angle between two intersecting line segments is invariant under a rotation then, as in the case of translations, any triangle, and so any polygon, is invariant under a rotation. As a closed curve may be considered to be the limit of a polygon as the number of sides is increased in a suitable way, it follows that any geometric figure remains invariant under a rotation.

Example 1. Show that the angle α between two line segments AB and BC is invariant under the rotational mapping

$$(x, y) \to (x \cos \theta - y \sin \theta,\ x \sin \theta + y \cos \theta)\,.$$

Solution: Let O be the origin of the co-ordinate system and, therefore, the centre of rotation.
Let A', B', C' be the image points of A, B, C respectively, $a = \angle ABC$ and $\alpha' = \angle A'B'C'$.



Using the results of Section 8.1 for the invariance of the lengths of line segments under a rotation, in $\triangle ABC$ and $\triangle A'B'C'$, we obtain

$$AB = A'B'\,,\ BC = B'C'\,,\ CA = C'A'\,,$$

$$\triangle ABC \equiv \triangle A'B'C'\,;$$

that is,

$$\angle\, ABC = \angle\, A'B'C'\,, \qquad \alpha = \alpha'\,.$$

Example 2. Find the equation of the image of the line $y = 2x - 1$ under the rotation $(x, y) \to (.6x - .8y,\ .8x + .6y) = (u, v)$. Illustrate the line and its image in a single-plane representation.

Solution: The transformation gives the equations

$$.6x - .8y = u\,, \qquad .8x + .6y = v$$

Hence,

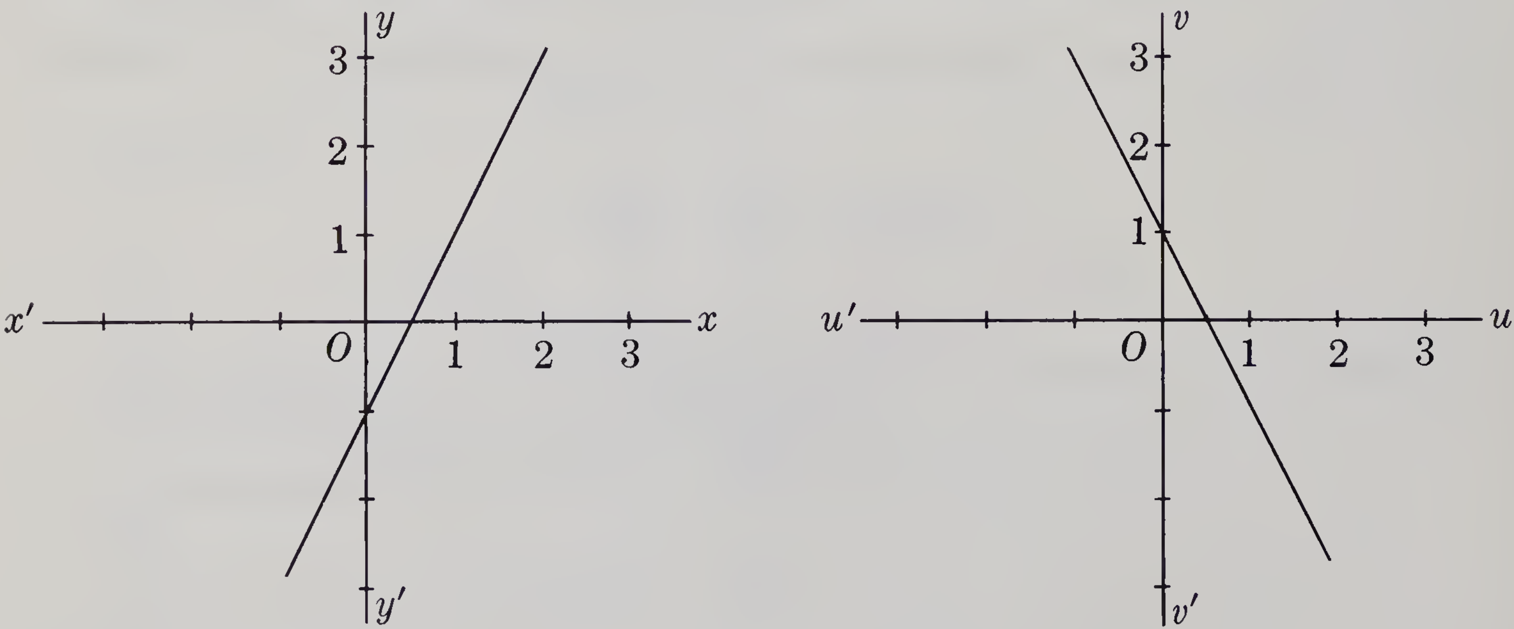
$$.36x + .64x = .6u + .8v$$

$$x = .6u + .8v \qquad \text{and} \qquad y = -.8u + .6v\,.$$

Therefore, the equation of the image line in the uv-plane is

$$(-.8u + .6v) = 2(.6u + .8v) - 1\,,$$
$$0 = 2u + v - 1\,.$$

In the two-plane representation, we have following diagrams.

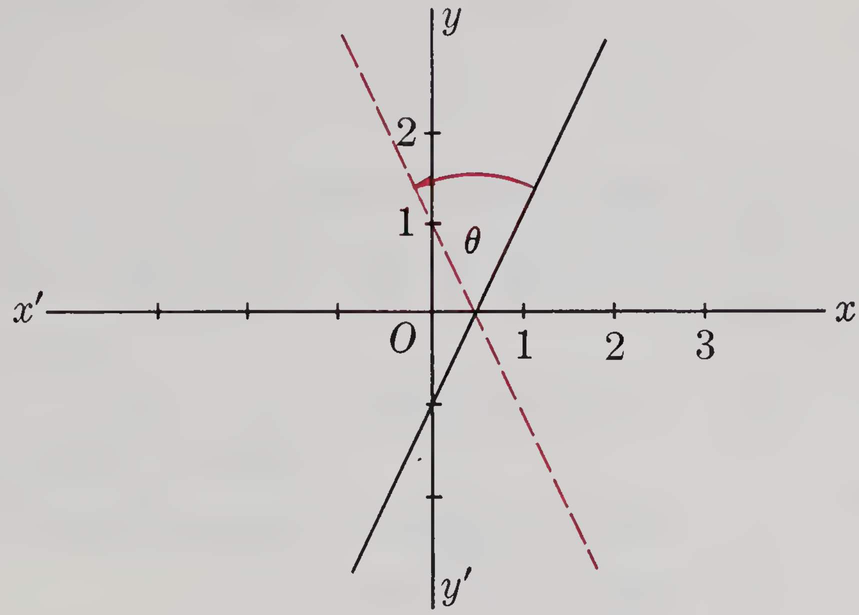


Original plane *Image plane*

In the single-plane representation, we allow the uv-axes and the xy-axes to coincide and have the same scale. In this case, the image line in the xy-plane has the equation

$$2x + y - 1 = 0\,.$$

The original and image lines and the angle of rotation are shown in the following diagram.



Example 3. Find the image of the curve determined by the equation

$$9x^2 + 4xy + 6y^2 = 20$$

under the rotation

$$(x,\, y) \to \left(\frac{x - 2y}{\sqrt{5}},\, \frac{2x + y}{\sqrt{5}}\right) = (u,\, v)\,.$$

Identify the curve and illustrate both the original and image curves on a single plane. What is the *angle of rotation* (to the nearest degree)? (By the *angle of rotation,* we mean the *angle between 0° and 360° through which all points of the original curve are rotated to produce the image curve.*)

Solution: The rotation determines

$$\begin{aligned} x - 2y &= \sqrt{5}u\,, \\ 2x + y &= \sqrt{5}v\,, \\ 4x + 2y &= 2\sqrt{5}v\,. \end{aligned}$$

Hence,

$$5x = \sqrt{5}u + 2\sqrt{5}v\,,$$

$$x = \frac{u + 2v}{\sqrt{5}}\,,$$

and

$$y = \frac{-2u + v}{\sqrt{5}}\,.$$

Therefore, the equation of the image curve is

$$9\left(\frac{u+2v}{\sqrt{5}}\right)^2 + 4\,\frac{(u+2v)\,(v-2u)}{5} + 6\left(\frac{v-2u}{\sqrt{5}}\right)^2 = 20\,,$$

$$9\,(u^2 + 4uv + 4v^2) + 4\,(-2u^2 + 2v^2 - 3uv) + 6\,(v^2 - 4uv + 4u^2) = 100\,,$$

$$(9 - 8 + 24)u^2 + (36 - 12 - 24)uv + (36 + 8 + 6)v^2 = 100\,,$$

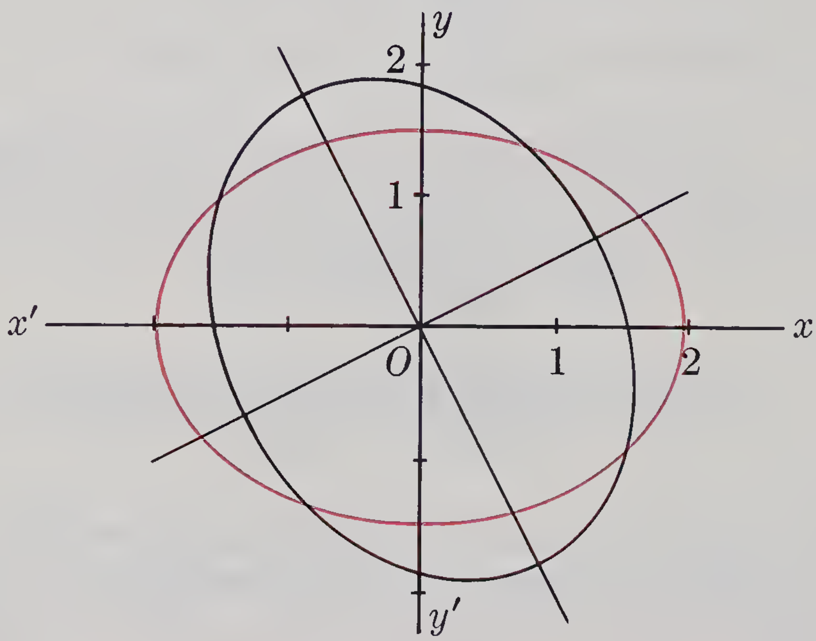
$$25u^2 + 50v^2 = 100\,,$$

$$u^2 + 2v^2 = 4\,,$$

$$\frac{u^2}{4} + \frac{v^2}{2} = 1\,.$$

The curve is an ellipse with centre at the origin, semi-major axis 2 units long and semi-minor axis $\sqrt{2}$ units long. In the single-plane representation, the equation of the image ellipse is

$$\frac{x^2}{4} + \frac{y^2}{2} = 1\,.$$



$$\cos\theta = \frac{1}{\sqrt{5}}$$

therefore,

$$\simeq .447;$$

$$\theta \simeq 63°.$$

The angle of rotation is approximately 63°.

EXERCISE 8.2

1. $A\,(\sqrt{3}, 1)$ and $B\,(\sqrt{3}, 3)$ are two points in the $x\ y$-plane. The rotation

$$(x,\,y) \to \left(\tfrac{\sqrt{3}}{2}x + \tfrac{1}{2}y\,,\ -\tfrac{1}{2}x + \tfrac{\sqrt{3}}{2}y\right)$$

is applied to the plane.

(a) Show that $\angle AOB$ is invariant under this rotation.

(b) Verify that all the sides and angles of $\triangle AOB$ are invariant under the rotation.

(c) Draw the original and image triangles in a single plane representation to common axes.

2. The four points $A(-\sqrt{2}, -\sqrt{2})$, $B(\sqrt{2}, -3\sqrt{2})$, $C(3\sqrt{2}, -\sqrt{2})$ and $D(\sqrt{2}, \sqrt{2})$ determine a quadrilateral.

(a) Find the image points under the rotation

$$(x, y) \to \left(\frac{x-y}{\sqrt{2}}, \frac{x+y}{\sqrt{2}}\right).$$

(b) Verify that both the original quadrilateral and the image quadrilateral are squares.

(c) Draw the original and image squares in a single plane representation to common axes.

3. The points $A(2, 0)$, $B(1, \sqrt{3})$, $C(-1, \sqrt{3})$, $D(-2, 0)$, $E(-1, -\sqrt{3})$, $F(1, -\sqrt{3})$ define a polygon.

(a) Find the image polygon under the rotation

$$(x, y) \to \left(\frac{-x-\sqrt{3}y}{2}, \frac{\sqrt{3}x-y}{2}\right).$$

(b) Find four other rotations that map $ABCDEF$ onto itself.

(c) Show that the polygon is a regular hexagon and indicate the symmetries of the rotations on a diagram.

In the following questions, find the equations of the images under the given rotation. Sketch the original and image figures on a single plane with the same set of axes. Identify the curve in each case.

4. $y = x$; $\quad (x, y) \to \left(\dfrac{x-y}{\sqrt{2}}, \dfrac{x+y}{\sqrt{2}}\right) = (u, v)$

5. $y = \sqrt{3}x$; $\quad (x, y) \to \left(\dfrac{\sqrt{3}x-y}{2}, \dfrac{x+\sqrt{3}y}{2}\right) = (u, v)$

6. $y = -x$; $\quad (x, y) \to (-y, x) = (u, v)$

7. $y - x = 4$; $\quad (x, y) \to \left(\dfrac{x-y}{\sqrt{2}}, \dfrac{x+y}{\sqrt{2}}\right) = (u, v)$

8. $y + x + \sqrt{2} = 0$; $\quad (x, y) \to \left(\dfrac{-x+y}{\sqrt{2}}, \dfrac{-x-y}{\sqrt{2}}\right) = (u, v)$

9. $y = 4$; $\quad (x, y) \to \left(\dfrac{3x-4y}{5}, \dfrac{4x+3y}{5}\right) = (u, v)$

10. $\left.\begin{aligned} 4x + 3y &= 5 \\ 3x - 4y &= -5 \end{aligned}\right\}$; $\quad (x, y) \to \left(\dfrac{3x+4y}{5}, \dfrac{3y-4x}{5}\right) = (u, v)$

11. $x^2 + y^2 = 4$; $\quad (x, y) \to \left(\dfrac{x - 2y}{\sqrt{5}}, \dfrac{2x + y}{\sqrt{5}}\right) = (u, v)$

12. $5x^2 - 6xy + 5y^2 = 32$; $\quad (x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right) = (u, v)$. Find the lengths of the semi-axes and the co-ordinates of the foci of the image figure.

13. $3x^2 + 3y^2 - 6xy = -32$; $\quad (x, y) \to \left(\dfrac{x + y}{\sqrt{2}}, \dfrac{-x + y}{\sqrt{2}}\right) = (u, v)$

14. $4y(y - \sqrt{3}x) + 3 = 0$; $\quad (x, y) \to \left(\dfrac{\sqrt{3}x + y}{2}, \dfrac{-x + \sqrt{3}y}{2}\right) = (u, v)$. Find the eccentricity and the coordinates of the foci of the image figure.

15. $x^2 - 2xy + y^2 = x + y$; $\quad (x, y) \to \left(\dfrac{x + y}{\sqrt{2}}, \dfrac{-x + y}{\sqrt{2}}\right) = (u, v)$.

16. $x^2 - y^2 = 4$; $\quad (x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right) = (u, v)$.

17. $xy = 4$; $\quad (x, y) \to \left(\dfrac{x + y}{\sqrt{2}}, \dfrac{-x + y}{\sqrt{2}}\right) = (u, v)$. Find the coordinates of the vertices and the foci of the image figure.

18. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$; $\quad (x, y) \to \left(\dfrac{4x - 3y}{5}, \dfrac{3x + 4y}{5}\right) = (u, v)$.

19. $x^2 = 2y$; $\quad (x, y) \to \left(\dfrac{\sqrt{3}x - y}{2}, \dfrac{x + \sqrt{3}y}{2}\right) = (u, v)$. Find the co-ordinates of the vertex in the image position.

20. $9x^2 - 24xy + 16y^2 = 20x + 15y$; $\quad (x, y) \to \left(\dfrac{3x - 4y}{5}, \dfrac{4x + 3y}{5}\right) = (u, v)$.

In each of the following questions, find the image of the region under the given rotation. Sketch the original and image regions on a single plane with the same set of axes.

21. $x < 2$; $\quad (x, y) \to \left(\dfrac{2x - 3y}{\sqrt{13}}, \dfrac{3x + 2y}{\sqrt{13}}\right) = (u, v)$.

22. $x > y$, $x \leq y + 1$; $\quad (x, y) \to \left(\dfrac{-5x - 12y}{13}, \dfrac{12x - 5y}{13}\right) = (u, v)$.

23. $x > -1 - y$, $x < -1 + y$, $y < 1$;
$(x, y) \to \left(\dfrac{-x + \sqrt{7}y}{2\sqrt{2}}, \dfrac{-\sqrt{7}x - y}{2\sqrt{2}}\right) = (u, v)$.

8.3. Simplification of Quadratic Relations by Rotations

In Example 3 of the previous section, the application of the rotation

$$(x, y) \to \left(\frac{x - 2y}{\sqrt{5}}, \frac{2x + y}{\sqrt{5}}\right)$$

reduced the equation

$$9x^2 + 4xy + 6y^2 = 20$$

to the simpler form

$$x^2 + 2y^2 = 4\,,$$

or

$$\frac{x^2}{4} + \frac{y^2}{2} = 1\,.$$

The latter form of the equation is the equation of an ellipse in standard form; the transformed ellipse has its major axis along the x-axis, its minor axis is along the y-axis, and its centre at the origin (0, 0).

Essentially the problem is that of transforming the general quadratic form

$$px^2 + qxy + ry^2$$

to the form of the sum or difference of two squares,

$$au^2 + bv^2\,, \qquad a, b \in Re\,.$$

The solution is not quite as simple as in the case of translations and so we have to proceed more formally.

Example 1. Determine a rotation which simplifies the equation

$$3x^2 - 2\sqrt{3}xy + y^2 = 36\,;$$

find the simplified equation and sketch the graphs of both equations to a common set of axes.

Solution: Apply the rotation

$$(x, y) \to (x \cos\theta - y \sin\theta,\ x \sin\theta + y \cos\theta) = (u, v)\,.$$

Then,

$$x \cos\theta - y \sin\theta = u\,,$$
$$x \sin\theta + y \cos\theta = v\,,$$

and, solving these two equations for x and y, we find

$$x = u \cos\theta + v \sin\theta$$

and

$$y = -u \sin\theta + v \cos\theta\,.$$

Substituting in the given equation, we obtain the transformed equation
$3(u\cos\theta + v\sin\theta)^2 - 2\sqrt{3}(u\cos\theta + v\sin\theta)(-u\sin\theta + v\cos\theta)$
$+ (-u\sin\theta + v\cos\theta)^2 = 36,$
or

$$\begin{aligned} &u^2(3\cos^2\theta + 2\sqrt{3}\sin\theta\cos\theta + \sin^2\theta) \\ &+ 2uv(3\sin\theta\cos\theta + \sqrt{3}\sin^2\theta - \sqrt{3}\cos^2\theta - \sin\theta\cos\theta) \\ &+ v^2(3\sin^2\theta + 2\sqrt{3}\sin\theta\cos\theta + \cos^2\theta) \\ &= 36\,. \end{aligned}$$

If the coefficient of uv is to be zero, which is the requirement for simplification, then

$$\begin{aligned} 2\sin\theta\cos\theta + \sqrt{3}(\sin^2\theta - \cos^2\theta) &= 0\,, \\ \sin 2\theta - \sqrt{3}\cos 2\theta &= 0\,, \\ \tan 2\theta &= \sqrt{3}\,. \end{aligned}$$

Any angle θ for which

$$\tan 2\theta = \sqrt{3}$$

will be a suitable rotation. Let us choose the smallest positive such angle.

Therefore,

$$\begin{aligned} 2\theta &= 60^\circ\,, \\ \theta &= 30^\circ\,. \end{aligned}$$

Hence,

$$\sin\theta = \tfrac{1}{2} \quad \text{and} \quad \cos\theta = \tfrac{\sqrt{3}}{2}\,.$$

The simplified transformed equation is, therefore,

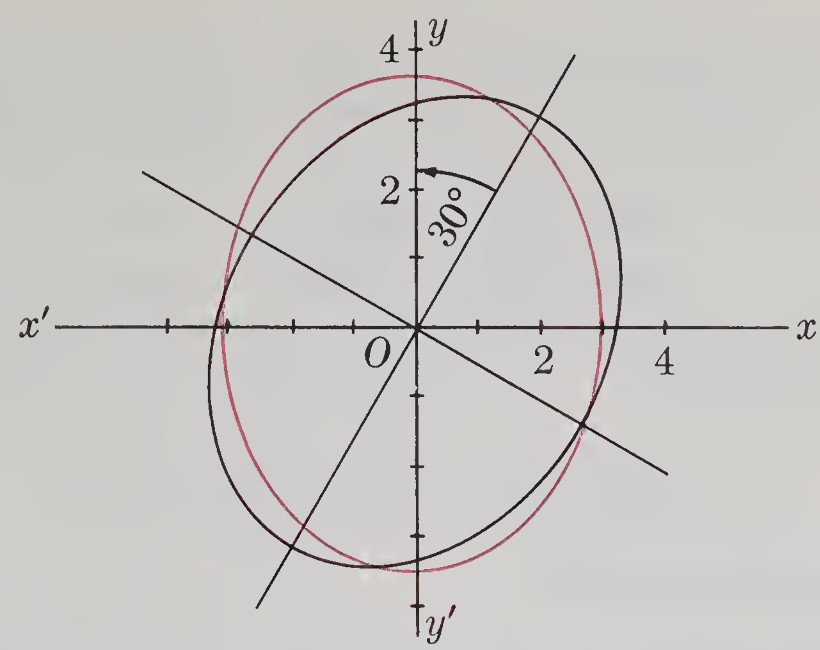
$$\begin{aligned} u^2(3\cdot\tfrac{3}{4} + 2\sqrt{3}\cdot\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{2} + \tfrac{1}{4}) + v^2(3\cdot\tfrac{1}{4} + 2\sqrt{3}\cdot\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{2} + \tfrac{3}{4}) &= 36 \\ u^2(\tfrac{9}{4} + \tfrac{6}{4} + \tfrac{1}{4}) + v^2(\tfrac{3}{4} + \tfrac{6}{4} + \tfrac{3}{4}) &= 36 \\ 16\,u^2 + 12\,v^2 &= 144 \\ \frac{u^2}{9} + \frac{y^2}{12} &= 1\,. \end{aligned}$$

Hence, the simplified equation in the xy-plane for a single-plane representation is

$$\frac{x^2}{9} + \frac{y^2}{12} = 1\,.$$

This curve, which is the original curve with all points rotated 30° counterclockwise, is an ellipse with centre at (0, 0), a semi-major axis of length $2\sqrt{3}$ units along the y-axis and a semi-minor axis of length 3 units along the x-axis.

As a rotation of $+30°$ transformed the original ellipse into the image ellipse, the original ellipse may be obtained from the image ellipse by a rotation of $-30°$, that is, by a clockwise rotation of the axes of the image ellipse through 30°.



Example 2. Simplify the equation $x^2 + 4xy + 4y^2 + 12x - 6y = 6\sqrt{5}$ to a standard form by suitable transformations. For both the original and image conics, find the co-ordinates of the focus or foci and of the vertex or vertices and find an equation of a directrix.

Solution: We can always bring a conic into a standard position, with focus or foci on or parallel to the x- or y-axis, by a rotation through an angle less than 90°, say θ. If we apply a rotation $(x, y) \to (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta) = (u, v)$, then

$$\begin{aligned} x &= u \cos\theta + v \sin\theta\,, \\ y &= -u \sin\theta + v \cos\theta\,, \end{aligned}$$

and the equation of the curve becomes

$$\begin{aligned} &(u \cos\theta + v \sin\theta)^2 + 4(u \cos\theta + v \sin\theta)(-u \sin\theta + v \cos\theta) \\ &+ 4(-u \sin\theta + v \cos\theta)^2 + 12(u \cos\theta + v \sin\theta) - 6(-u \sin\theta + v \cos\theta) \\ &= 6\sqrt{5}\,, \end{aligned}$$

$$\begin{aligned} &u^2(\cos^2\theta - 4 \sin\theta \cos\theta + 4 \sin^2\theta) \\ &+ uv\,(2 \sin\theta \cos\theta + 4 \cos^2\theta - 4 \sin^2\theta - 8 \sin\theta \cos\theta) \\ &+ v^2\,(\sin^2\theta + 4 \sin\theta \cos\theta + 4 \cos^2\theta) \\ &+ u(12 \cos\theta + 6 \sin\theta) + v(12 \sin\theta - 6 \cos\theta) \\ &= 6\sqrt{5}. \end{aligned}$$

The coefficient of $uv = 0$, if

$$-6 \sin\theta \cos\theta + 4(\cos^2\theta - \sin^2\theta) = 0\,;$$

that is, if

$$\tan 2\theta = +\tfrac{4}{3}\,.$$

Hence, 2θ is a first-quadrant angle and

$$\cos 2\theta = +\frac{3}{\sqrt{3^2 + 4^2}} = +\tfrac{3}{5}\,.$$

($0 < \theta < 90°$; therefore, $0 < 2\theta < 180°$ and $\sin 2\theta > 0$.)

Therefore,

$$2\cos^2\theta - 1 = +\tfrac{3}{5} \quad \text{and} \quad 1 - 2\sin^2\theta = +\tfrac{3}{5}.$$

Thus,

$$\cos^2\theta = \tfrac{4}{5} \quad \text{and} \quad \sin^2\theta = \tfrac{1}{5},$$

$$\cos\theta = \frac{2}{\sqrt{5}} \quad \text{and} \quad \sin\theta = \frac{1}{\sqrt{5}},$$

and the rotation required is

$$(x, y) \to \left(\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y,\ \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right).$$

Therefore, the equation becomes

$$u^2\left(\tfrac{4}{5} - 4 \cdot \tfrac{1}{\sqrt{5}} \cdot \tfrac{2}{\sqrt{5}} + 4 \cdot \tfrac{1}{5}\right) + v^2\left(\tfrac{1}{5} + 4 \cdot \tfrac{1}{\sqrt{5}} \cdot \tfrac{2}{\sqrt{5}} + 4 \cdot \tfrac{4}{5}\right) + u\left(12 \cdot \tfrac{2}{\sqrt{5}} + 6 \cdot \tfrac{1}{\sqrt{5}}\right)$$
$$+ v\left(12 \cdot \tfrac{1}{\sqrt{5}} - 6 \cdot \tfrac{2}{\sqrt{5}}\right) = 6\sqrt{5},$$

that is

$$\tfrac{25}{5}v^2 + \tfrac{30}{\sqrt{5}}u = 6\sqrt{5}$$
$$v^2 = -\frac{6}{\sqrt{5}}u + \frac{6\sqrt{5}}{5}$$
$$v^2 = -\frac{6}{\sqrt{5}}(u - 1).$$

Thus, if we now apply the translation

$$(u, v) \to (u - 1, v) = (s, t),$$

we obtain one of the standard forms of the parabola

$$t^2 = -\frac{6}{\sqrt{5}}s.$$

We can illustrate the three curves on a single-plane representation if we note that

$$x^2 + 4xy + 4y^2 + 12x - 6y = 6\sqrt{5}$$

is reduced by the rotation

$$(x, y) \to \left(\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y,\ \frac{1}{\sqrt{5}}x + \frac{2}{\sqrt{5}}y\right)$$

to

$$y^2 = -\frac{6}{\sqrt{5}}(x - 1)$$

which is in turn reduced by the translation

$$(x, y) \to (x - 1, y)$$

to

$$y^2 = -\frac{6}{\sqrt{5}}x.$$

In the st-plane, the focus is $(-\frac{3}{2\sqrt{5}}, 0)$, the vertex $(0, 0)$, and the directrix the line whose equation is $s = \frac{3}{2\sqrt{5}}$.
Observe that

$$u = s + 1, \qquad v = t.$$

In the uv-plane, the focus is $(1 - \frac{3}{2\sqrt{5}}, 0)$, the vertex $(1, 0)$, and the directrix the line whose equation is $u = \frac{3}{2\sqrt{5}} + 1$.
Note that

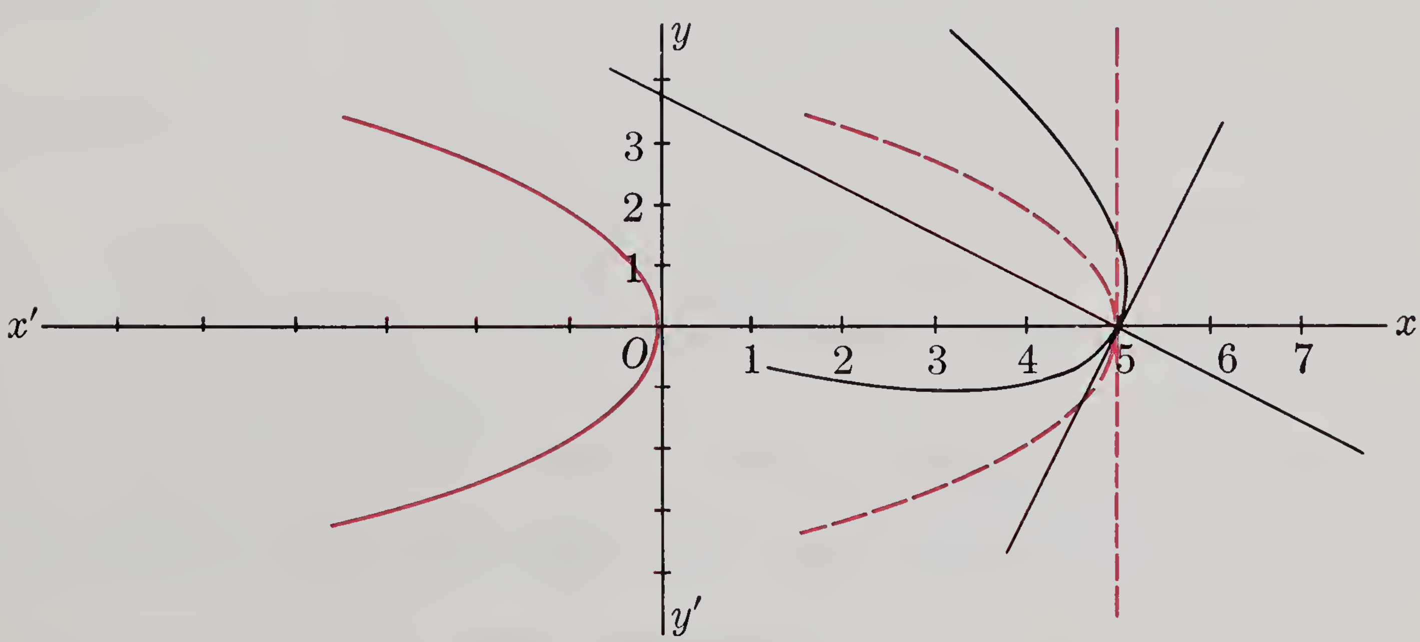
$$x = \frac{2u + v}{\sqrt{5}}, \qquad y = \frac{-u + 2v}{\sqrt{5}}.$$

In the xy-plane, the focus is

$$\left(\frac{2 - \frac{3}{\sqrt{5}}}{\sqrt{5}}, \frac{-1 + \frac{3}{2\sqrt{5}}}{\sqrt{5}}\right) = \left(\frac{2\sqrt{5} - 3}{5}, \frac{-2\sqrt{5} + 3}{10}\right).$$

The vertex in the xy-plane is $(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$. In the xy-plane, the directrix is the line whose equation is

$$\frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y = \frac{3}{2\sqrt{5}} + 1, \qquad \text{or} \qquad 4x + 2y = 3 + 2\sqrt{5}.$$



EXERCISE 8.3

In questions (1) to (10), find a suitable rotation which eliminates the xy-term in the given equation. Sketch the original and image curve to the same set of axes in each case.

1. $x^2 - 2xy + y^2 = 12x + 12y$
2. $x^2 + 2\sqrt{3}y - y^2 = 4$
3. $3x^2 + 4\sqrt{3}xy - y^2 = 9$
4. $17x^2 + 16xy + 17y^2 = 225$
5. $x^2 - 4xy + y^2 = 16$
6. $xy = 9$
7. $3x^2 + 10xy + 3y^2 = 25$
8. $25x^2 - 14xy + 25y^2 = 288$
9. $x^2 - 2xy + y^2 - 2x - 2y + 1 = 0$
10. $x^2 + 2\sqrt{3}xy + 3y^2 + 4\sqrt{3}x + 4y = 0$

Reduce each of the following equations to a suitable standard form for a conic.

11. $x^2 + 4xy + y^2 + 6x - 12y = 0$
12. $5x^2 + 6xy + 5y^2 + 6x - 12y + 21 = 0$
13. $x^2 - 2xy + y^2 + 5x = 0$
14. $x^2 + 6x - 3y + 6 = 0$
15. $3x^2 - 2\sqrt{3}xy + y^2 + 4x - 4\sqrt{3}y = 0$
16. In questions (11) to (15), find the lengths of the semi-axes and the co-ordinates of the vertices of the conic in standard position. Find the co-ordinates of the vertices of the original conic.
17. (a) Show that if the rotation
$$(x, y) \to (x \cos \theta - y \sin \theta, x \sin \theta + y \cos \theta) = (u, v)$$
is applied to
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$
the coefficient of the uv term is
$$(a - b) \sin 2\theta + 2h \cos 2\theta.$$
(b) Find the condition on 2θ which will make the coefficient of uv in (a) equal to zero.
18. Use the method suggested by 17(b) in question (15) to find $\sin \theta$ and $\cos \theta$.
19. If $a = b$ in 17(b), find the value of 2θ and hence of θ. Use this conclusion to do question (1).

Chapter Summary

The mapping $(x, y) \to (x \cos \theta - y \sin \theta,\ x \sin \theta + y \cos \theta)$ · Rotations · Invariance of lengths of line segments and of angle under rotational mappings · Simplification of equations for quadratic relations by rotational mappings

REVIEW EXERCISE 8

1. Find the images of the points $M(2, -1)$, $N(3, 4)$, $P(0, 5)$, $Q(-1, -1)$, under the rotations.

   (a) $(x, y) \to \left(\dfrac{\sqrt{3}x - y}{2}, \dfrac{x + \sqrt{3}y}{2}\right)$

   (b) $(x, y) \to (-x, -y)$

   (c) $(x, y) \to \left(\dfrac{2x - y}{\sqrt{5}}, \dfrac{x + 2y}{\sqrt{5}}\right)$

In each case, state the angle of rotation and show each point and its image in a single-plane representation.

2. Suppose that the origin, $A(5, 0)$, and $B(4, 3)$ are vertices of a triangle OAB. Find the co-ordinates of the vertices of a triangle $OA'B'$ which is the image of OAB under a rotation through an angle of 60°.

3. Find an equation for the line which is the image of $y = x$ after the plane has been rotated through 45°. Show both original and image lines in a single-plane representation.

Find the equations of the images under the following rotations. In each case, identify the curve and sketch both original and image figures in a single-plane representation.

4. $x + y + 5 = 0;\quad (x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right).$

5. $y = \dfrac{3}{2};\quad (x, y) \to \left(\dfrac{4x - 3y}{5}, \dfrac{3x + 4y}{5}\right)$

6. $x^2 + y^2 = 9;\quad (x, y) \to \left(\dfrac{x - 3y}{\sqrt{10}}, \dfrac{3x + y}{\sqrt{10}}\right)$

7. $xy = 8;\quad (x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right)$

8. $\dfrac{x^2}{4} - \dfrac{y^2}{25} = 1;\quad (x, y) \to (-y, x)$

A point (x_1, y_1) is inside the ellipse with equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

if and only if

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} < 1.$$

If a plane is rotated, points inside an ellipse remain inside and points outside remain outside. Transform the following ellipses into standard form and determine whether the given points are inside the ellipse.

9. $3x^2 - 2xy + 3y^2 = 8$, $(0, \sqrt{2})$, $(-\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2})$

10. $66x^2 + 24xy + 59y^2 = 500$, $(\frac{-9}{5}, \frac{-13}{5})$ $(\frac{-11}{5}, \frac{-2}{5})$

In the following questions, find a suitable rotation that eliminates the xy term in the given equation. In each case, sketch the original and image curves in a single-plane representation.

11. $5x^2 + 4xy + 5y^2 - 9 = 0$

12. $xy = -32$

13. $4x^2 + 12xy + 9y^2 = 52$

14. $4x^2 - 20xy + 25y^2 - 15x - 6y = 0$

15. Using the method we developed in question (18), of Exercise 8.3, repeat question (13).

16. (a) Simplify the equation
$$6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$$
by first applying a suitable rotation to remove the xy-term and then applying a suitable translation to remove the first-degree terms in x and y.
(b) Simplify the equation in part (a) by first applying a translation to eliminate the first-degree terms in x and y. (*Hint:* Let the translation be $u = x + h$, $v = y + k$. Express the conditions for the vanishing of the first-degree terms in u and v as equations in h and k.) Then apply a rotation to eliminate the xy-term.
(c) Identify the graph of the relation. Show the original and all image graphs in (a) and (b) in a single-plane representation.

17. For the image of the mapping in question (16), find the lengths of the semi-axes and the foci of the conic involved in the transformation.

18. Remove the xy-term from the equation
$$6x^2 + 3xy + 2y^2 - 39 = 0$$
by a suitable rotation. Identify the conic and find its eccentricity and the co-ordinates of the foci in the image position. Find the co-ordinates of the foci in the original position.

Chapter 9

REFLECTIONS AND DILATATIONS

9.1. Reflections of Points and Polygons

In the two previous chapters, we have considered mappings that could be described in terms of translations and rotations. Under these mappings, distances and angles remain invariant, as does the order of the points on a geometric figure, such as, for example, a polygon or a curve.

There are other mappings under which distances, angles or both remain invariant; an interesting example is the *reflection* transformation. An example of a reflection transformation is given by

$$(x, y) \to (-x, y) .$$

If we consider the images of three points

$$A\,(1, 2), \qquad B\,(3, 4), \text{ and } C\,(2, 5)$$

under such a transformation, we see that the image points are

$$A'\,(-1, 2), \qquad B'\,(-3, 4), \text{ and } C'\,(-2, 5) .$$

The two triangles ABC and $A'B'C'$ are shown in Figure 9.1.

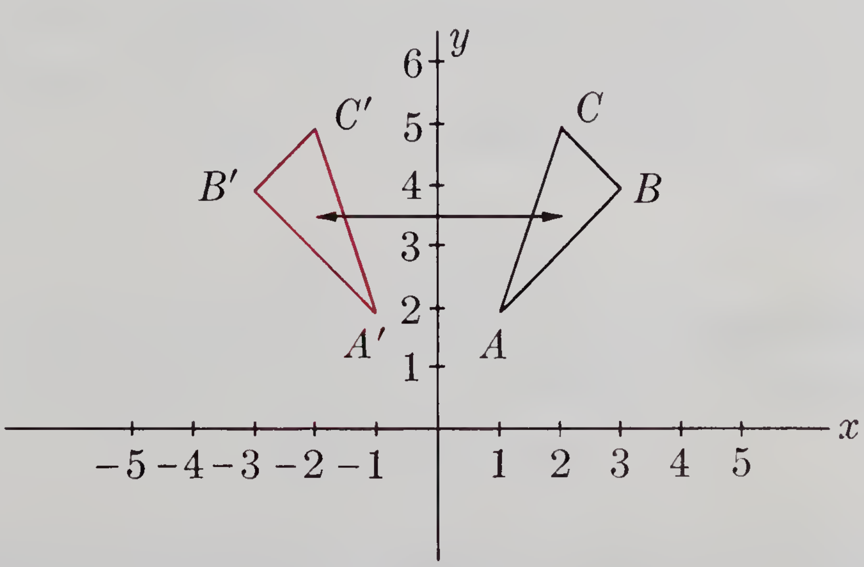


Figure 9.1

We easily establish that

$$AB^2 = (3 - 1)^2 + (4 - 2)^2 = 4 + 4 = 8\,,$$

and

$$A'B'^2 = (-3 + 1)^2 + (4 - 2)^2 = 4 + 4 = 8\,,$$

so that

$$AB = A'B'\,.$$

Also,

$$\text{slope of } AB = m_1 = \frac{4 - 2}{3 - 1} = 1\,,$$

$$\text{slope of } AC = m_2 = \frac{5 - 2}{2 - 1} = 3\,;$$

hence,

$$\tan \angle BAC = \frac{3 - 1}{1 + 3} = \tfrac{1}{2}\,.$$

and similarly

$$\tan \angle B'A'C' = \tfrac{1}{2}\,.$$

Therefore,

$$\angle BAC = \angle B'A'C'\,.$$

However, although the lengths of line segments and the sizes of angles are preserved under this transformation, there is an important difference from the translations and rotations. *The directions are changed.* Thus, if we proceed around $\triangle ABC$ from A to B to C back to A (i.e., counterclockwise) the interior is on the left, while for $\triangle A'B'C'$ the interior is on the right if we proceed from A' to B' to C' and back to A' (i.e. clockwise). Thus, we may say that the order of the vertices is reversed by a reflection. The reader should check that this reversal of order does not occur under translations or rotations.

If we use an analogue from geometrical optics and consider a plane mirror to be placed along the y-axis perpendicular to the page, then the reflection or mirror image of $\triangle ABC$ appears to be $\triangle A'B'C'$. This analogy is the reason for the name of this type of transformation.

The transformation

$$(x, y) \to (-x, y)$$

is a reflection in the y-axis.

The transformation

$$(x, y) \to (x, -y)$$

is another reflection transformation and this time the axis of reflection is the x-axis as shown in Figure 9.2.

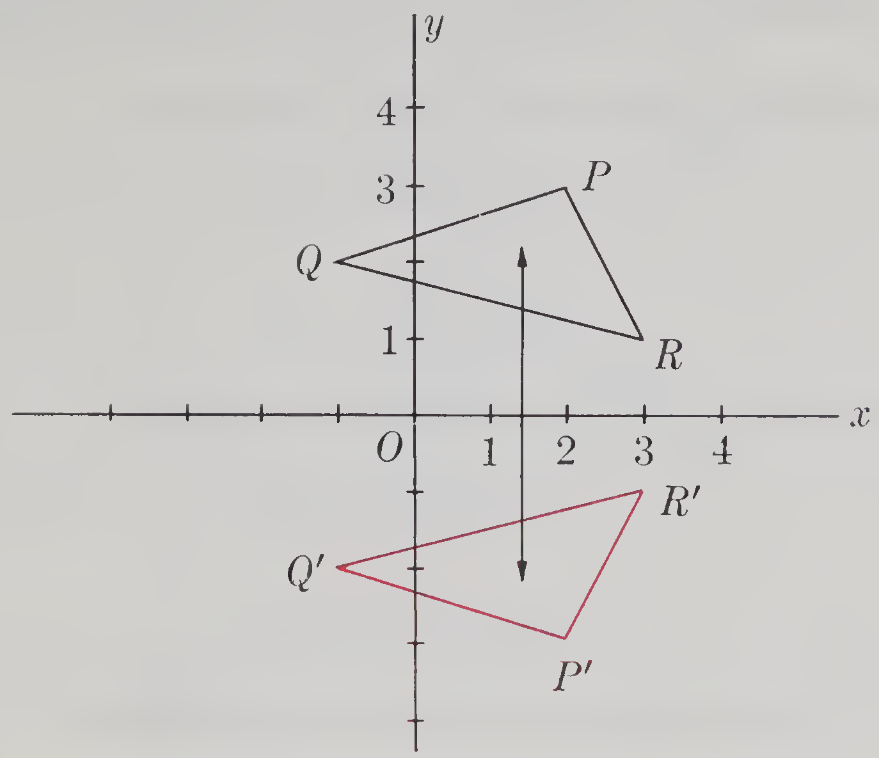


Figure 9.2

Again the order of the points $P'Q'R'$ is the reverse of the order of PQR, if we traverse the triangles so that the interior is on the left in each case.

Note that, in Figure 9.1,

$$\triangle A'B'C' \text{ is the image of } \triangle \text{ABC}$$

and

$$\triangle ABC \text{ is the image of } \triangle A'B'C'$$

under the reflection

$$(x, y) \to (-x, y) .$$

In Figure 9.2,

$$\triangle P'Q'R' \text{ is the image of } \triangle PQR$$

and

$$\triangle PQR \text{ is the image of } \triangle P'Q'R'$$

under the reflection

$$(x, y) \to (x, -y) .$$

This reciprocity between image and original is not true in general for translations and rotations. It is, however, true for the rotation

$$(x, y) \to (-x, -y) ,$$

that is, the rotation through 180° or π radians. Is it true for any other rotation? Or any translation?

The rotation

$$(x,\ y) \to (-x,\ -y)$$

may be viewed as a double reflection or as a reflection in a point, though this latter is a mathematical "reflection" and does not have a physical analogue.

Figure 9.3 (a) shows the double reflection, the reflection

$$(x,\ y) \to (-x,\ y)$$

followed by

$$(x,\ y) \to (x,\ -y)\ .$$

Figure 9.3 (b) shows the double reflection, the reflection

$$(x,\ y) \to (x,\ -y)$$

followed by

$$(x,\ y) \to (-x,\ y)\ .$$

In each case, the result is equivalent to the transformation

$$(x,\ y) \to (-x,\ -y)\ .$$

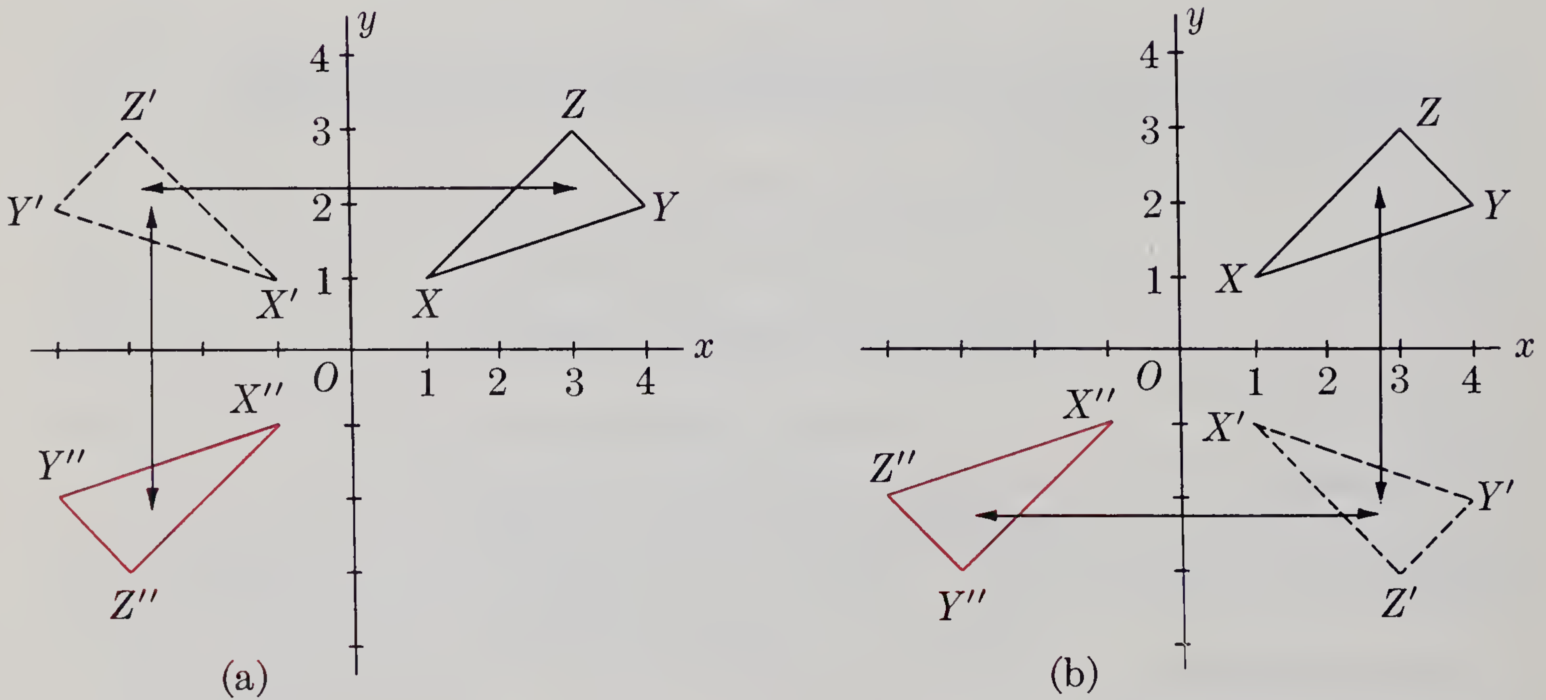


Figure 9.3

When we consider the transformation

$$(x,\ y) \to (-x,\ -y)$$

as a mathematical reflection in the origin, we are defining a reflection in the origin as follows:

Given the point P and the origin O, let us extend the line OP beyond O to P' so that $PO = OP'$. Then P' is the reflection of P in the origin. Figure 9.4 illustrates the reflection of $\triangle XYZ$ in the origin, the image being $\triangle X''Y''Z''$.

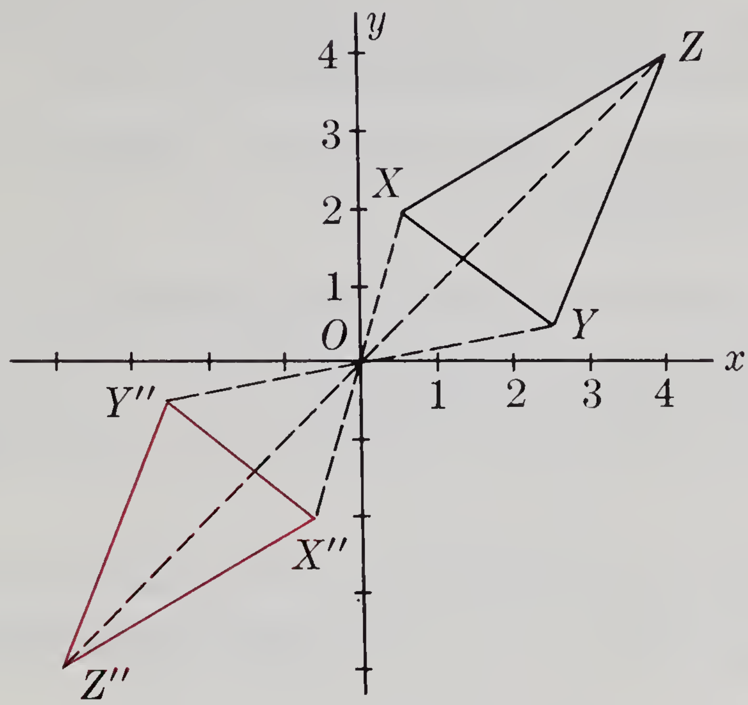


Figure 9.4

We note that the double reflection, or a reflection in the origin, leaves the interior of the triangle on the same side (the left in this case) of the paths XYZ and $X''Y''Z''$. Each of the two reflections reverses the order (see Figure 9.3) and the final result of the two reflections is the preservation of the order. This case must hold, as the result is equivalent to that obtained by a rotation.

EXERCISE 9.1

1. (a) Find the images of the three points $P(3, 2)$, $Q(7, 3)$, and $R(4, 0)$ under the transformation
$$(x, y) \rightarrow (x, -y)\,.$$
   (b) Sketch the original and image triangles.
   (c) Prove that the triangles are congruent.
   (d) What is the equation of the axis of reflection for this transformation?

2. (a) Find images A', B', C', of the three points $A(3, 5)$, $B(1, 2)$, and $C(8, 1)$ by a reflection in the origin. Plot the six points.
   (b) Show that $\angle ABC = \angle A'B'C'$.
   (c) Check to show that the order of the vertices in triangles ABC and $A'B'C'$ has been preserved by this reflection. Contrast this result with the result in question (1). Explain.

3. (a) Find the co-ordinates of A', B', and C', the images of $A(4,8)$, $B(4,2)$, and $C(7,2)$ under the reflection transformation $(x, y) \to (-x + 6, y)$. Plot the six points.

   (b) What is the axis of reflection for this mapping?

4. (a) Find images corresponding to the points $M(1, 0)$, $N(2, 1)$, and $P(7, 4)$ under the reflection transformation $(x, y) \to (y, x)$.

   (b) Plot the six points. Sketch the axis of reflection for this transformation and state its equation.

5. (a) Under the transformations F and F followed by G, where $F : (x, y) \to (x, -y)$ and G $(x, y) \to (-x, y)$, find a first and second set of images for the points $A(1, 9)$, $B(3, 6)$, and $C(5, 3)$.

   (b) Show that the collinearity of $A, B,$ and C is preserved in each transformation.

   (c) What are the axes of reflection for the transformations in (a)?

   (d) What single reflection transformation is equivalent to the two reflections in (a)?

6. (a) Under the transformations given by S and S followed by T, where $S : (x, y) \to (x, -y)$ and $(x, y) \to (-x - 2, y)$, find a final set of images for the points A, B, and C of question (5).

   (b) What is the axis of reflection for the second reflection in 6(a)?

   (c) Use mapping notation to describe a single transformation equivalent to the two reflections in (a).

   (d) In which point could the transformation in (c) be considered as a reflection?

7. (a) Under the transformations V followed by W and W followed by V, where $V : (x, y) \to (x + 1, y - 3)$ and $W : (x, y) \to (-x, -y)$, find final sets of images of $A(3, 1)$, $B(-6, -1)$, and $C(2, -4)$.

   (b) Show that, if A, B, and C are collinear and B is between A and C, B remains between A and C after a reflection in an axis or the origin.

9.2. Reflections of Lines and Curves

The invariance of the lengths of line segments and angles under reflection implies the invariance of polygons, and consequently of all geometric loci, including curves under reflection. However, we must remember that the order of the points on a curve is reversed by each reflection. We shall now examine the reflections of some lines and conics.

When we wished to find the effect of translations or rotations on the equations of various curves we used a two-plane representation at first. After the equation had been transformed from one in x and y to one in u and v, we returned to a single-plane representation.

If we do the same for the reflection

$$(x, y) \to (-x, y) = (u, v)\,,$$

we see that

$$u = -x\,, \quad v = y\,,$$

and so that

$$x = -u\,, \quad y = v$$

is the required substitution in a given equation in x and y. For example, the line given by the equation

$$x + y = 1 \qquad \text{in the } xy\text{-plane}$$

becomes the line given by the equation

$$-u + v = 1 \qquad \text{in the } uv\text{-plane.}$$

Therefore, in the single plane representation, the image line is given by

$$-x + y = 1\,.$$

Thus, we can pass from the original equation to the image equation by replacing x by $-x$ and y by y without using the two plane representation first. The reader should contrast this situation with the cases of translation and rotation.

Example 1.

(a) Apply the reflections,

$$\text{(i)} \quad R_1 : (x, y) \to (-x, y)$$

$$\text{(ii)} \quad R_2 : (x, y) \to (x, -y)$$

to the line

$$3x + 4y = 12\,.$$

(b) Find the result of the reflection R_1 followed by R_2 and show that the result is that given by the reflection

$$R_3 : (x, y) \to (-x, -y)\,.$$

(c) Show that R_3 is also the result of the reflection R_2 followed by R_1. Show the various lines on one Cartesian plane.

Solution:

(a) (i) Under the reflection

$$R_1 : (x, y) \to (-x, y) ,$$

x is replaced by $-x$ and y by y; that is, y is unchanged. Therefore, the line

$$l = \{(x, y) \mid 3x + 4y = 12\} ,$$

under the reflection R_1, becomes

$$\begin{aligned} l_1 &= \{(x, y) \mid 3(-x) + 4y = 12\} \\ &= \{(x, y) \mid -3x + 4y = 12\} . \end{aligned}$$

(ii) Similarly, the line

$$l = \{(x, y) \mid 3x + 4y = 12\} ,$$

under the reflection R_2,
becomes the line

$$\begin{aligned} l_2 &= \{(x, y) \mid 3x + 4(-y) = 12\} \\ &= \{(x, y) \mid 3x - 4y = 12\} . \end{aligned}$$

(b) If the reflection R_2

is applied to the line l_1, it becomes

$$\begin{aligned} l_3 &= \{(x, y) \mid -3x + 4(-y) = 12\} \\ &= \{(x, y) \mid -3x - 4y = 12\} . \end{aligned}$$

If the reflection

$$R_3 : (x, y) \to (-x, -y)$$

is applied to the line l, it becomes the line

$$\begin{aligned} \{(x, y) \mid 3(-x) + 4(-y) = 12\} &= \{(x, y) \mid -3x - 4y = 12\} \\ &= l_3 . \end{aligned}$$

(c) Similarly, the reflection R_1 applied to the line l_2 gives the line

$$\begin{aligned} \{(x, y) \mid 3(-x) - 4y = 12\} &= \{(x, y) \mid -3x - 4y = 12\} \\ &= l_3 . \end{aligned}$$

We note that we could have used a two-plane representation by writing

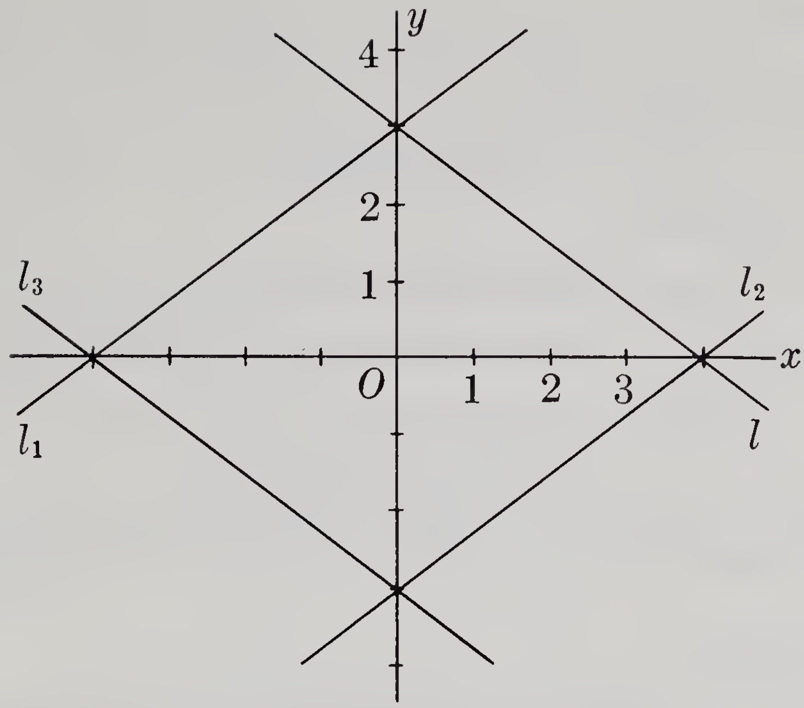
$$R_1 : (x, y) \to (-x, y) = (u, v) ,$$

so that the reflection of the line

$$l = \{(x, y) \mid 3x + 4y = 5\} \text{ in the } xy\text{-plane}$$

becomes the line

$$l_1' = \{(u, v) \mid -3u + 4v = 5\} \text{ in the } uv\text{-plane} .$$



Example 2. Find the image of

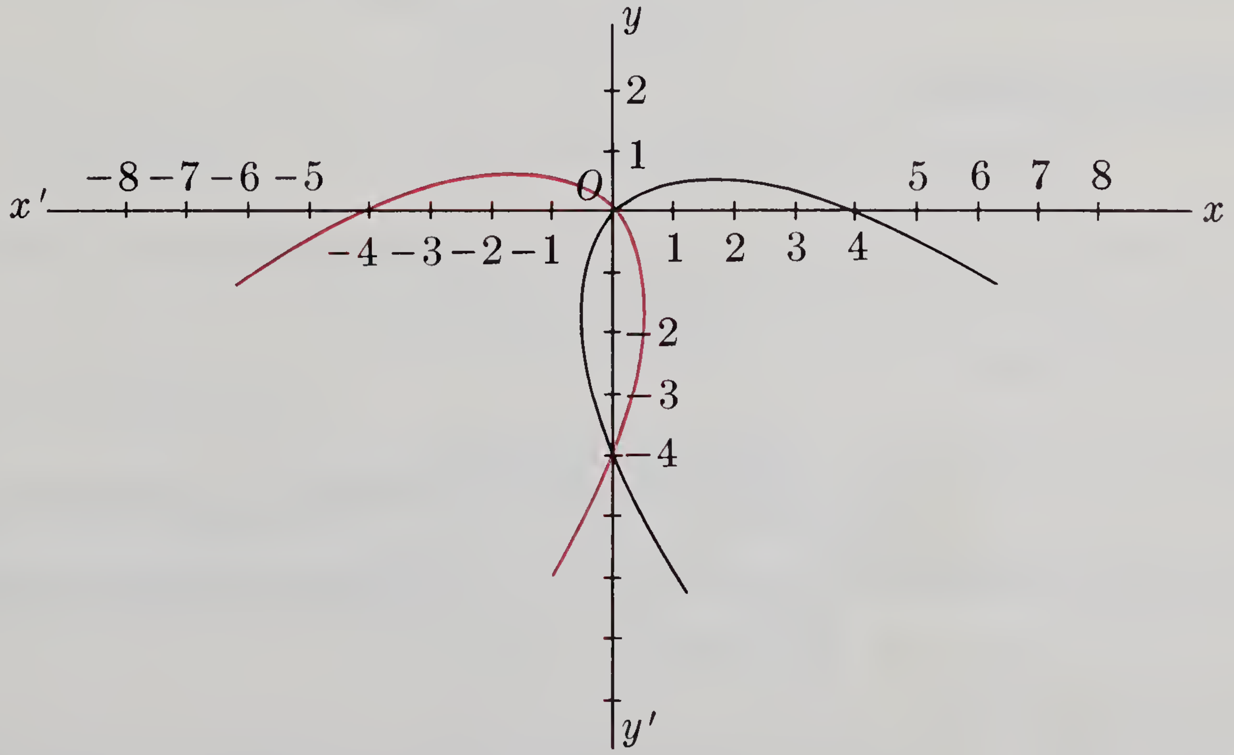
$$x^2 + 2xy + y^2 = 4(x - y)$$

under the reflection

$$(x, y) \to (-x, y) .$$

Solution: Under the reflection, x is replaced by $-x$ and y by y, so that the image equation is given by

$$(-x)^2 + 2(-x)y + y^2 = 4(-x - y) ,$$
$$x^2 - 2xy + y^2 = -4(x + y) .$$



EXERCISE 9.2

1. Apply the reflection
$$R_1 : (x, y) \to (-x, y)$$
to the line whose equation is
$$2x + 3y = -6 ,$$
and apply the reflection
$$R_2 : (x, y) \to (x, -y)$$
to the image under R_1. Show that the final result is also obtained by applying the reflection
$$R_3 : (x, y) \to (-x, -y)$$
to the original line. Show the original lines and the two image lines in a single-plane representation.

2. Under the reflection
$$R : (x, y) \to (-x, -y)$$
find the image of the point of intersection of the lines whose equations are
$$7x - 3y = -21 \quad \text{and} \quad 3x + 6y = 8 .$$
Sketch the two original and two image lines in a single-plane representation.

3. A circle has its centre at $(0, 5)$ and radius 5 units. Find the equation of the image circle under the reflection
$$R : (x, y) \to (x, -y)$$
Show both circles in one Cartesian plane.

4. Find the region that is the image of the region defined by
$$2x + 3y < -6$$
under the reflection
$$R : (x, y) \to (-x, y) .$$

5. Find the region which is the image of the region defined by
$$x^2 + y^2 \leq 16, \ 3x + 5y \geq 15$$
under the reflection
$$R : (x, y) \to (x, -y) .$$
Sketch the original region and its image in a single-plane representation.

6. Find defining equations for the region that is the image of the region defined by
$$x + y > 2, \ x + y < 6 ,$$
under the reflection
$$R : (x, y) \to (-x, -y) .$$
Show both regions on one Cartesian plane.

7. (a) Find the image of the hyperbola $x^2 - y^2 = 16$ under the reflections
   (i) $R_1 : (x, y) \rightarrow (-x, y)$,
   (ii) $R_2 : (x, y) \rightarrow (x, -y)$,
   (iii) $R_3 : (x, y) \rightarrow (-x, -y)$.
   (b) Find the image of the hyperbola $xy = 8$ under the reflections (i) R_1, (ii) R_2, (iii) R_3.
   (c) How many different hyperbolas are involved in part (b)? Sketch them on one Cartesian plane.

8. Find the equation of, and sketch the image of the ellipse
$$\frac{(x+4)^2}{16} + \frac{y^2}{9} = 1$$
under the reflection
$$R : (x, y) \rightarrow (-x, y) .$$

9. (a) Find the equation of the image of the parabola
$$y^2 = 4(x-2)$$
   under the reflection
$$R : (x, y) \rightarrow (-x+2, y) ,$$
   (b) Sketch the original parabola and its image in a single-plane representation.
   (c) State the equation of the axis of reflection.

10. (a) Apply in turn the reflections
$$R_1 : (x, y) \rightarrow (y, x) ,$$
$$R_2 : (x, y) \rightarrow (y, -x) ,$$
   to the ellipse
$$\frac{(x-12)^2}{144} + \frac{y^2}{25} = 1 .$$
   (b) How many *different* ellipses are involved? Sketch them in a single-plane representation. Show on the graph the axes of reflection.

11. Find the image of the graph of
$$y = \frac{6}{x^2+2}$$
under the reflection
$$R : (x, y) \rightarrow (x, -y) .$$
Show both original and image on one Cartesian plane.

12. Find the image of the region
$$y > \frac{6}{x^2+2}, \qquad x^2 + y^2 < 9 ,$$
under the transformation in question (11). Sketch the original region and its image on one Cartesian plane.

9.3. Dilatations of Polygons

In all the transformations we have studied so far, lengths and angles have been invariant. We shall now consider some elementary transformations under which lengths or angles or both are not invariant.

Example 1. Find the image points of

$$A(1, 1) \quad B(-1, 1) \quad \text{and} \quad C(0, -1)$$

under the transformation

$$(x, y) \to (3x, 3y) .$$

Show the original and image points on one Cartesian plane.

Solution:

The point $A(1, 1)$ has the image $A'(3, 3)$.
The point $B(-1, 1)$ has the image $B'(-3, 3)$.
The point $C(0, -1)$ has the image $C'(0, -3)$.

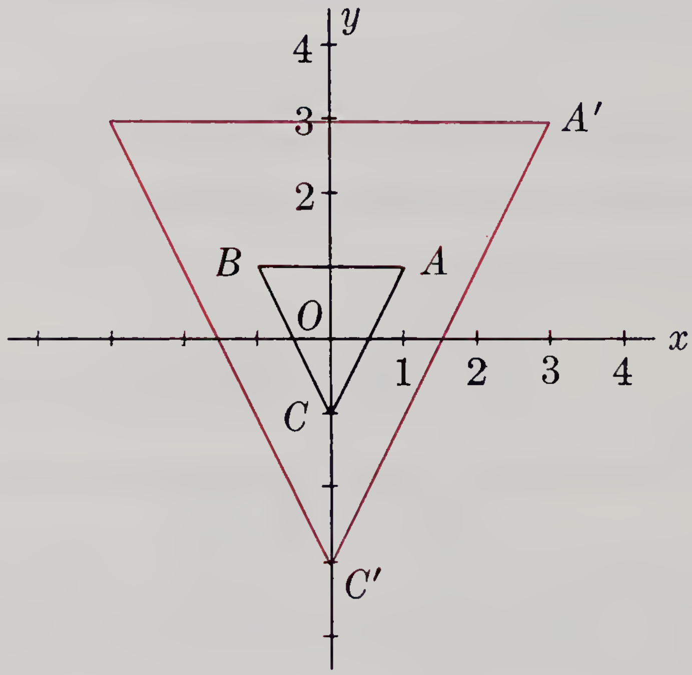


Figure 9.5

If we form the triangles ABC and $A'B'C'$ in Figure 9.5, we see immediately that they are similar. Thus, the transformation

$$(x, y) \to (3x, 3y)$$

preserves the angles of the triangles; that is, the angles are invariant under the transformation. Also, the length of AB is 2 units and of $A'B'$ 6 units, so that the ratio of the sides of the triangles is

$$\frac{A'B'}{AB} = \frac{B'C'}{BC} = \frac{C'A'}{CA} = \frac{3}{1} .$$

It would appear, therefore, that the coefficient of x or y (when the coefficients are equal) gives the amplification of the transformation. Such a transformation is called a homogeneous isotropic dilatation, which means that the amplitude of the stretching is the same at all points of the plane and in all directions.

The general homogeneous isotropic dilatation transformation is given by

$$(x,y) \longrightarrow (ax, ay), \quad a \in Re^{+}$$

and a is the amplification of the dilatation.

A more general transformation, which changes lengths, is

$$(x, y) \longrightarrow (ax, by), \qquad a, b \in Re^{+}, \qquad a \neq b\,.$$

Such a transformation also changes angles in general, as we can show in the following examples.

Example 2. Find the images, A' and B', of $A(2,4)$ and $B(-4,2)$ under the transformation

$$(x, y) \longrightarrow (\tfrac{1}{2}x, \tfrac{3}{2}y)\,.$$

Show the original and image points on one Cartesian plane. Show that

$$\angle AOB \neq \angle A'OB',$$

where O is the origin of the co-ordinate system.

Solution:

The image of $A(2,4)$ is $A'(1,6)$.
The image of $B(-4,2)$ is $B'(2,3)$.

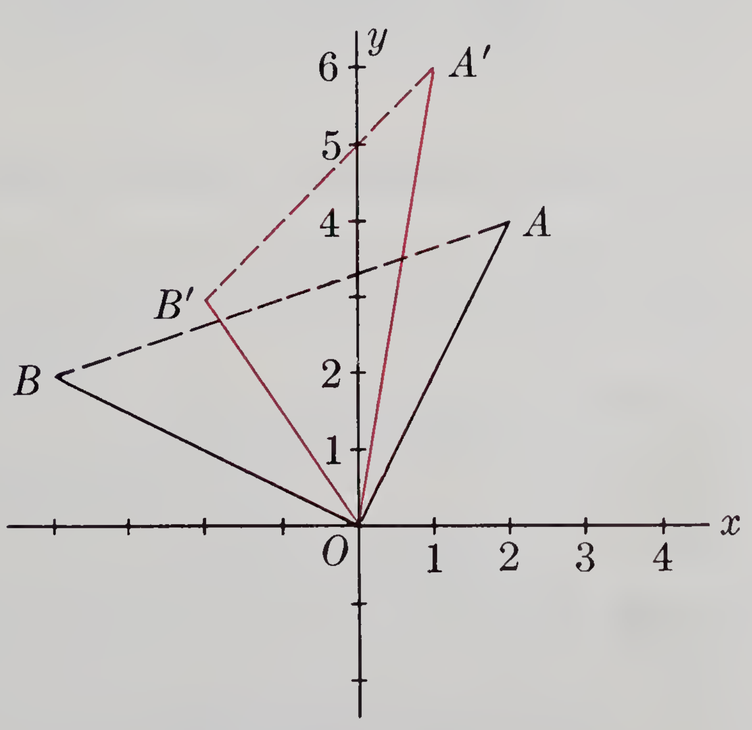


Figure 9.6

The slope of $OA = m_1 = 2$ and the slope of $OB = m_2 = -\frac{1}{2}$. The numbers m_1 and m_2 are negative reciprocals; therefore,

$$\angle AOB = 90°,$$

and the slope of $OA' = m_1' = 6$, and the slope of $OB' = m_2' = -\frac{3}{2}$.
Thus,

$$\begin{aligned}\tan \angle A'OB' &= \frac{-6 + (-\frac{3}{2})}{1 + 6(-\frac{3}{2})} \\ &= \frac{-6 - 1.5}{1 - 9} \\ &= \frac{7.5}{8} \\ &= .9375 .\end{aligned}$$

Hence,

$$\angle A'OB' \simeq 43° .$$

Therefore, $\angle AOB \neq \angle A'OB'$.

We note that some angles are invariant; for example, if C is the point (2, 0) and D the point (0, 2), then the images are $C'(1, 0)$ and $D'(0, 3)$, so that

$$\angle COD = \angle C'OD' = 90° .$$

If, in Figure 9.6, we form the triangles AOB and $A'OB'$, we see that the two triangles are neither congruent nor similar. We should expect this result, as the transformation has changed both lengths and angles. However, the two figures are both triangles. In general under dilatations, an n-gon (a polygon of n sides) remains an n-gon with no change in n, the number of sides.

Example 3. Using the points of Example 2, find the ratios $OA : OA'$, $OB : OB'$ and $AB : A'B'$.

Solution:

$$\begin{aligned}OA^2 &= 2^2 + 4^2 \\ &= 4 + 16 \\ &= 20 , \\ OA &= \sqrt{20} .\end{aligned} \qquad\qquad \begin{aligned}OA'^2 &= 1^2 + 6^2 \\ &= 1 + 36 \\ &= 37, \\ OA' &= \sqrt{37} .\end{aligned}$$

Therefore,

$$OA : OA' = \sqrt{20} : \sqrt{37} .$$

$$\begin{aligned} OB^2 &= (-4)^2 + 2^2 \\ &= 16 + 4 \\ &= 20\,, \\ OB &= \sqrt{20}\,. \end{aligned} \qquad\qquad \begin{aligned} OB'^2 &= (-2)^2 + 3^2 \\ &= 4 + 9 \\ &= 13\,, \\ OB' &= \sqrt{13}\,. \end{aligned}$$

Therefore,

$$OB : OB' = \sqrt{20} : \sqrt{13}\,.$$

$$\begin{aligned} AB^2 &= (-4-2)^2 + (2-4)^2 \\ &= 6^2 + 2^2 \\ &= 36 + 4 \\ &= 40\,, \\ AB &= \sqrt{40}\,. \end{aligned} \qquad\qquad \begin{aligned} A'B'^2 &= (-2-1)^2 + (3-6)^2 \\ &= 3^2 + 3^2 \\ &= 9 + 9 \\ &= 18\,, \\ A'B' &= \sqrt{18}\,. \end{aligned}$$

Therefore,

$$AB : A'B' = \sqrt{40} : \sqrt{18}\,.$$

The reader should verify that, if C is the point $(3, 6)$ and C' its image, then $OC : OC' = \sqrt{20} : \sqrt{37}$ and that O, A and C, are collinear and so are O, A', and C'. This result illustrates the fact that the ratio of the lengths of line segments in a given direction is independent of the length of the line segments. The ratio of the lengths of line segments in any given direction is called the *amplification* in that direction.

The dilatation

$$(x, y) \to (ax, by), \qquad a, b \in Re^+, \quad a \neq b$$

is homogeneous but not isotropic; that is, the amplification is the same at every point in the plane but not the same in all directions.

EXERCISE 9.3

1. (a) The points $A\,(1, 1)$, $B\,(-1, 1)$, $C\,(-1, -1)$, and $D\,(1, -1)$ are vertices of a square. Find image points for these vertices under the homogeneous, isotropic dilatation

$$D : (x, y) \to (4x, 4y)\,.$$

   (b) Calculate slopes to show that

$$\angle AOD = \angle A'OD'\,.$$

   (c) Repeat (b) by showing that O, A, A' and O, D, D' are collinear.

2. Repeat question (1) for a congruent square centred at $E\,(6, 8)$ rather than at $O\,(0, 0)$.

3. (a) Find the image of the square in question (1) under the homogeneous, nonisotropic dilatation

$$D_1 : (x, y) \rightarrow (2x, \tfrac{1}{2}y) .$$

(b) Sketch the original square and its image in a single-plane representation.

(c) Identify the image figure. Show that

$$\angle AOD \neq \angle A'OD' .$$

4. (a) The vertices of an isosceles triangle are $A(1, 0)$, $B(-1, 0)$, and $C(0, -\sqrt{3})$. Find the vertices of the image triangle under the dilatation,

$$D : (x, y) \rightarrow (\sqrt{3}x, \sqrt{3}y) .$$

(b) Sketch both triangles in one Cartesian plane.

(c) Prove that the triangles ABC and $A'B'C'$ are similar.

5. (a) Find the image of the triangle in question 4(a) under the dilatation

$$D_1 : (x, y) \rightarrow \left(2x, \frac{y}{3}\right).$$

(b) Sketch the original triangle and its image.

(c) Show that the new image triangle $A''B''C''$ is isosceles but not similar to $\triangle ABC$.

6. The points $M(2, 2)$, $O(0, 0)$, $N(-3, 3)$, $P(-3, -2)$, and $Q(2, -2)$ are the vertices of a pentagon. Find the co-ordinates of the vertices of the image of this pentagon under the dilatation

$$D : (x, y) \rightarrow (3x, 3y) .$$

Sketch both pentagons in a single plane representation.

7. (a) Find the image of the original pentagon in question (6) under the dilatation

$$D_1 : (x, y) \rightarrow (3x, 2y)$$

(b) Sketch the original pentagon and the image under D_1 in a single-plane representation.

(c) Show that $MQ \parallel M''Q''$ and $OM \neq OM''$.

9.4. Dilatations of Lines and Curves

In the previous section and exercise, we observed that the number of sides of a polygon was unchanged under a dilatation but that the shape was changed. As a consequence a line remains a line under a dilatation, but one curve may be changed into a curve of a different type. For example, a circle can be changed into an ellipse. However, a closed curve (*e.g.*, an ellipse) cannot be changed into an open curve (*e.g.*, a parabola or hyperbola) by a finite dilatation.

Example 1. Find the equation of the image of the line $3x + 4y = 6$ under the transformation

$$(x, y) \longrightarrow (3x, 2y) = (u, v)\,.$$

Draw a two-plane diagram and a single-plane diagram of the original line and its image.

Solution: For the transformation,

$$3x = u \quad \text{and} \quad 2y = v\,;$$

therefore,

$$x = \tfrac{1}{3}u \quad \text{and} \quad y = \tfrac{1}{2}v\,.$$

Substituting for x and y in

$$3x + 4y = 6,$$

we obtain the transformed equation

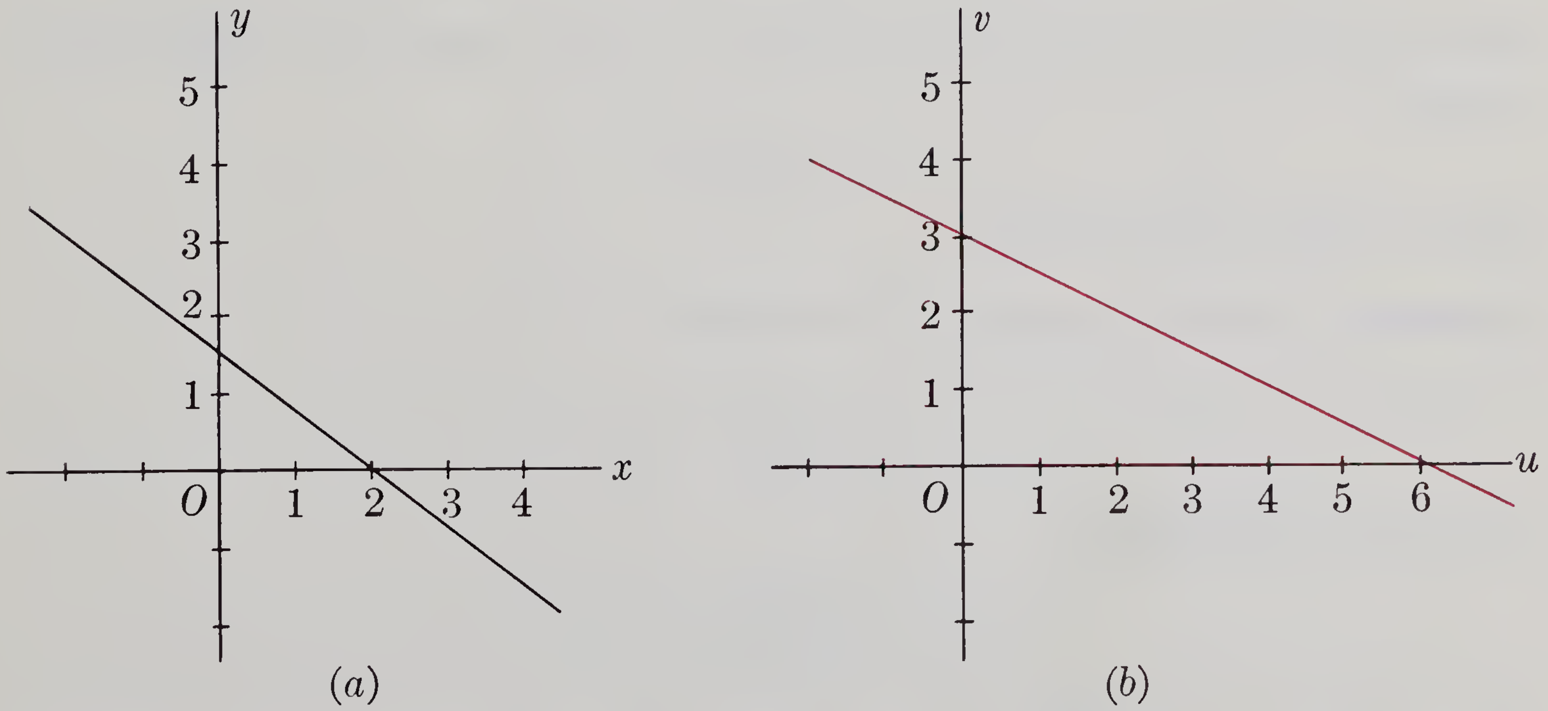
$$3\left(\tfrac{1}{3}u\right) + 4\left(\tfrac{1}{2}v\right) = 6\,.$$

Hence, the equation of the image line is

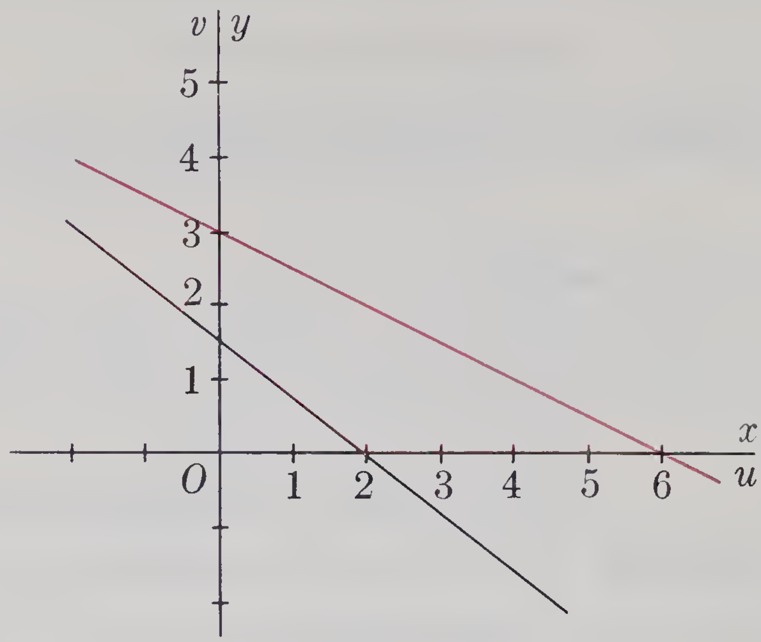
$$u + 2v = 6\,.$$

In the single-plane representation, the uv- and xy-axes coincide, and we may write the image equation as

$$x + 2y = 6\,.$$



Two-Plane Representation



Single-Plane Representation

To verify the correctness of these equations, we may find the intercept points of the original line and the images of these points and check that the images do lie on the image line. The line

$$3x + 4y = 6$$

passes through $(2, 0)$ and $(0, \frac{3}{2})$. The image points are $(6, 0)$ and $(0, 3)$ respectively. The line

$$u + 2v = 6$$

passes through $(6, 0)$ and $(0, 3)$ as required, or, in the single plane representation, the line

$$x + 2y = 6$$

passes through $(6, 0)$ and $(0, 3)$.

Example 2. Find the equation of the image of

$$\frac{x^2}{9} + \frac{y^2}{4} = 1$$

under the transformation

$$(x, y) \to \left(\frac{x}{3}, \frac{y}{2}\right) = (u, v)\,.$$

Identify both the original and image curves. Draw a two-plane diagram and a single-plane diagram.

Solution: For the transformation

$$\frac{x}{3} = u \quad \text{and} \quad \frac{y}{2} = v \, ;$$

hence,

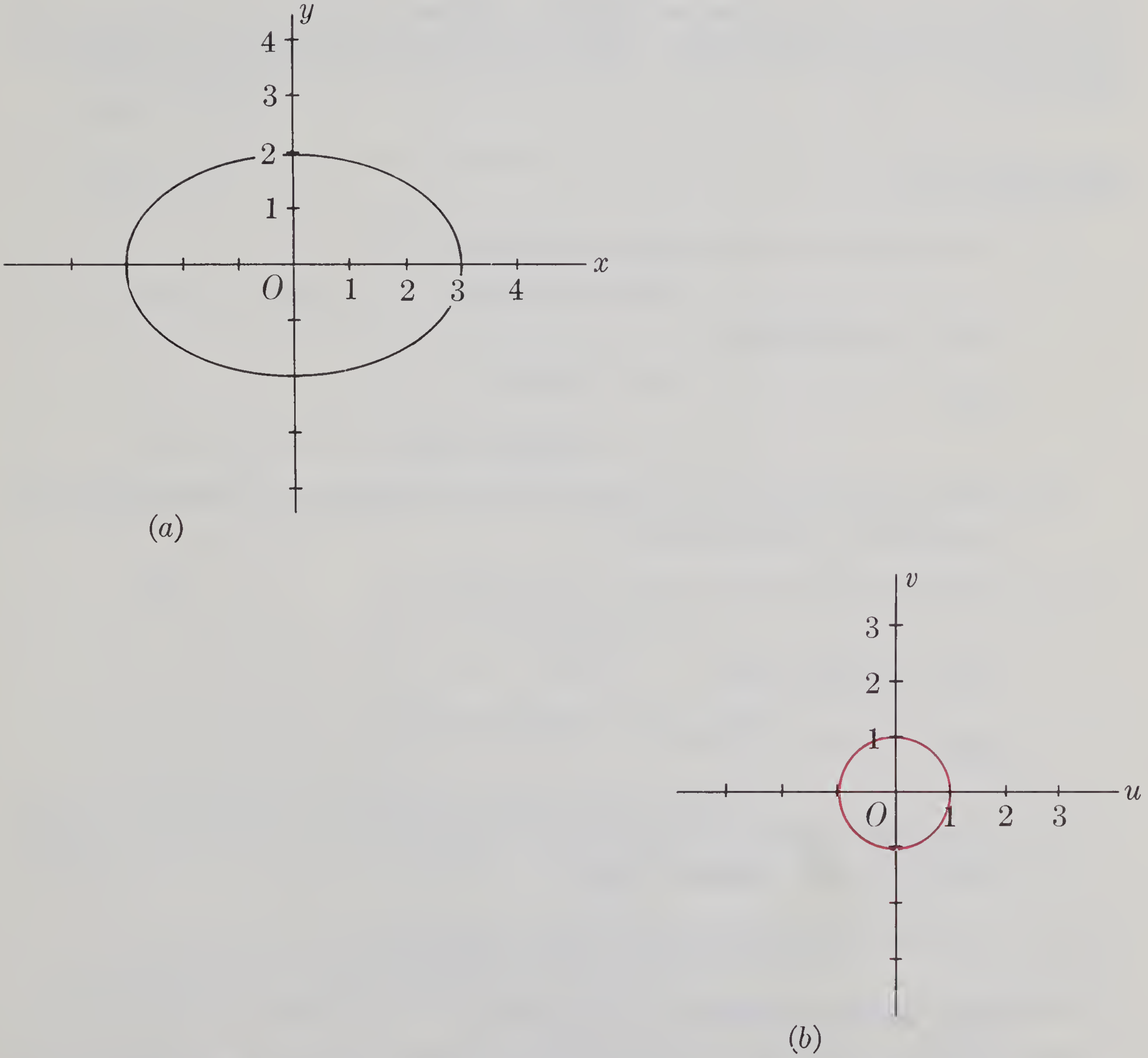
$$x = 3u \quad \text{and} \quad y = 2v \, .$$

Therefore the equation becomes, in the uv-plane,

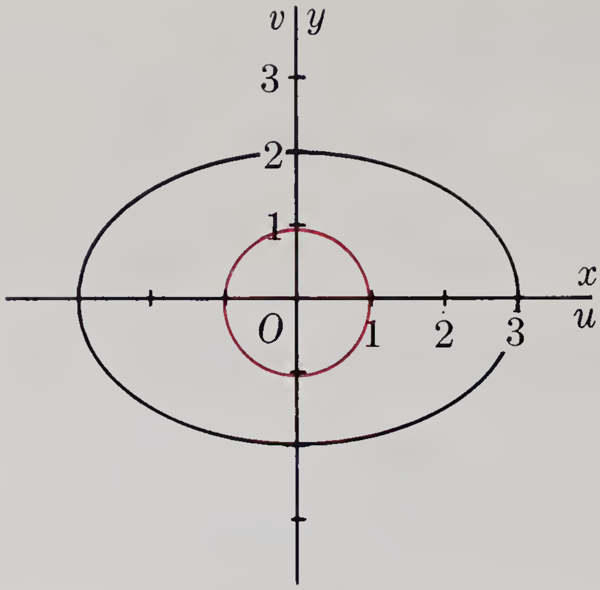
$$\frac{(3u)^2}{9} + \frac{(2v)^2}{4} = 1 \, ,$$

$$\frac{9u^2}{9} + \frac{4v^2}{4} = 1 \, ,$$

$$u^2 + v^2 = 1 \, .$$



Two-Plane Representation



Single-Plane Representation

The original curve is an ellipse centred at O with semi-major axis 3 units on the x-axis and semi-minor axis 2 units. The image curve is a circle of unit radius with centre at O.

EXERCISE 9.4

1. (a) Find equations of the images of the line

$$x + y + 1 = 0$$

under the dilatations

$$D_1 : (x, y) \rightarrow (2x, 2y) = (u, v)$$

and

$$D_2 : (x, y) \rightarrow (2x, 4y) = (s, t) .$$

(b) Sketch the three lines involved in both a three-plane representation and a single-plane representation.

2. (a) Find the image of the circle whose equation is

$$x^2 + y^2 = 1$$

under the dilatations

$$D_1 : (x, y) \rightarrow (3x, 3y) = (u, v)$$

and

$$D_2 : (x, y) \rightarrow (3x, 5y) = (s, t) .$$

(b) Sketch the original circle and the two images of (a) in both a three-plane and single-plane representation.

3. (a) Find the image of the circle whose equation is

$$x^2 - 2x + y^2 + 6y = -1$$

under the dilatation D_1 of question (2).

(b) Find the centre of the image circle and the image of the centre of the original circle of (a).

4. (a) Find the image of the parabola

$$x^2 = -y$$

under the dilatation

$$D : (x, y) \to (\tfrac{1}{2}x, \tfrac{1}{5}y) = (u, v) .$$

Sketch the original and image graphs in one Cartesian plane.

(b) Find the co-ordinates of the focus of both the original parabola and its image.

5. Find the equation of the image of

$$\frac{x^2}{49} + \frac{y^2}{576} = 1$$

under the transformation

$$(x, y) \to (x, \tfrac{7}{24}y) = (u, v) .$$

Identify both original and image curves and sketch them in a single-plane representation.

6. Repeat question (5) but precede the transformation of question (5) by the translation

$$T : (x, y) \to (x - 8, y + 4) .$$

7. (a) Find the images of the hyperbola whose equation is

$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$

under each of the transformations

$$D_1 : (x, y) \to \left(x, \frac{y}{2}\right) = (u, v)$$

and

$$D_2 : (x, y) \to \left(\frac{x}{3}, \frac{y}{4}\right) = (m, n) .$$

(b) Sketch the three curves involved in a single-plane representation.

8. (a) Translate the hyperbola whose equation is

$$\frac{(x + 4)^2}{25} - \frac{(y - 3)^2}{49} = -1$$

so that the image hyperbola is centred at the origin.

(b) Find the image of the transformed hyperbola in (a) under the dilatation

$$(x, y) \to \left(2x, \frac{y}{7}\right) .$$

9. (a) Find the image of the ellipse
$$2x^2 + \sqrt{3}xy + y^2 = 5$$
under the rotation mapping
$$(x, y) \to \left(\frac{\sqrt{3}}{2}x - \frac{1}{2}y, \frac{1}{2}x + \frac{\sqrt{3}}{2}y\right).$$
(b) To the transformed ellipse in (a) apply the dilatation
$$(x, y) \to \left(\frac{x}{2\sqrt{2}}, \frac{y}{\sqrt{10}}\right) = (u, v) .$$
Find the lengths of the semi-axes of the resulting curve.
(c) Show the three ellipses involved in a single-plane representation.

10. (a) Apply a suitable transformation to the conic represented by
$$6x^2 + 24xy - y^2 - 12x + 26y + 11 = 0$$
to remove the xy-term.
(b) To the transformed hyperbola in part (a) apply the translation
$$(x, y) \to (x - \tfrac{1}{5}, y + \tfrac{7}{5}) .$$
(c) Identify the conic. Find its eccentricity.
(d) To the conic obtained in (b) apply the dilatation
$$(x, y) \to \left(\frac{x}{\sqrt{2}}, \frac{y}{\sqrt{3}}\right).$$
Describe the image. Find its eccentricity.

11. Find the image of the graph of
$$y = \frac{-5}{x^2 + 1}$$
under the transformation
$$(x, y) \to \left(\frac{x}{2}, y\right).$$
Sketch both graphs in a single-plane representation.

Chapter Summary

Reflection transformations · Invariance of length and angle · Change in direction under a reflection · Reflections of lines and conics · Dilatation transformations · Change in length and/or angle · Amplitude of a dilatation.
General homogeneous isotropic dilatation
$$(x, y) \to (ax, ay), \qquad a \in Re^+ .$$
General homogeneous dilatation
$$(x, y) \to (ax, by), \qquad a, b \in Re^+, a \neq b .$$
Dilatations of lines and conics.

REVIEW EXERCISE 9

1. (a) Find images for the points $A(4, 1)$, $B(6, 5)$, and $C(9, -1)$ under the transformation
$$(x, y) \to (-x, y)\,.$$
(b) Prove that $\angle BAC = \angle B'A'C'$.

2. (a) Find the image of the graph of
$$y = |x|$$
under the reflection
$$(x, y) \to (x, -y)\,.$$
(b) Sketch the original and image graphs in a single-plane representation.
(c) Use the three points $(4, 4)$, $(0, 0)$ and $(-2, 2)$ with their images to show that order has been reversed by this reflection.

3. Find the images of the points $P(-1, 5)$, $Q(0, 3)$, and $R(6, 7)$ under the transformation
$$(x, y) \to (x, -y -2)\,.$$
Sketch the six points involved. State the axis of reflection for this transformation.

4. Under the reflection
$$(x, y) \to (-x, -y)\,,$$
find the image of triangle ABC where A is $(2, -1)$, B is $(1, -3)$, and C is $(6, -5)$. Sketch the two triangles in a single-plane representation and show that they are congruent by calculating lengths of corresponding sides.

5. Find the image of the region defined by
$$x - y + 1 < 0, \quad x + 5y - 5 > 0, \quad x < 5$$
under the reflection
$$(x, y) \to (x, -y)\,.$$
Sketch both regions in a single-plane representation.

6. Find the image of the region defined by
$$4x^2 + y^2 \leq 36, \quad 2x + y \geq 6$$
under the transformation
$$(x, y) \to (-x, -y)\,.$$
Show both regions in one Cartesian plane.

7. Find the equation of the image of the conic
$$9x^2 - 16y^2 = 144$$
under the transformation
$$(x, y) \to (x, -y -8)\,.$$
Sketch both original and image in a single-plane representation. State the equation of the axis of reflection for this transformation.

8. (a) Sketch the equilateral hyperbola $xy = 8$ and its reflection in the x-axis both on one Cartesian plane.
   (b) Sketch the circle $x^2 + y^2 = 8$ on the same set of axes.
   (c) Shade in the region defined by
$$x^2 + y^2 \geq 8, \qquad |xy| \leq 8\,.$$

9. The points $A\,(0, 4)$, $B\,(0, -4)$, $C\,(6, -4)$, and $D\,(6, 4)$ are vertices of a rectangle. Find the vertices of a rectangle which is the image of the original rectangle under the dilatation
$$(x, y) \to (\tfrac{3}{2}x, \tfrac{3}{2}y)$$
Show that the rectangles are similar.

10. (a) Find the image of the rectangle in (9) under the dilatation
$$(x, y) \to \left(\frac{x}{2}, 2y\right)$$
   (b) Prove that the original rectangle and its new image are not similar.
   (c) Sketch the three rectangles of questions (9) and (10) in one Cartesian plane.

11. Repeat question (9) if the original rectangle is first moved to the left so that its centre is at $(-3, 0)$. Show that the angle between the diagonals remains constant under the homogeneous, isotropic dilatation of (9).

12. Find the vertices of the image of an isosceles triangle with vertices at
$$P\,(5, 0),\ Q\,(5, -6), \text{ and } R\,(9, -3)$$
under the dilatation
$$(x, y) \to (3x, 3y)\,.$$
Sketch the two triangles in a single-plane representation. Show that the image triangle is isosceles and similar to the original.

13. Find the image of the isosceles triangle in (12) under the dilatation
$$(x, y) \to \left(\frac{x}{2}, \frac{y}{3}\right).$$
Show that the image triangle is isosceles but not similar to the original.

14. (a) Find the images of an ellipse whose equation is
$$4x^2 + 25y^2 = 100$$
   under the dilatations
$$D_1 : (x, y) \to \left(\frac{x}{2}, \frac{y}{2}\right) = (u, v)\,,$$
$$D_2 : (x, y) \to \left(\frac{x}{5}, \frac{y}{2}\right) = (u, v)\,.$$
   (b) Show the original ellipse and its images in a single-plane representation.

15. Find an equation for the image of the conic

$$x^2 - y^2 = -25$$

under the dilatation

$$(x, y) \to (4x, 4y) = (u, v)$$

Describe the image conic and sketch both original and image in a single-plane representation.

16. Repeat question (15) if the dilatation is

$$(x, y) \to \left(\frac{x}{2}, \frac{y}{3}\right) = (u, v) .$$

17. By a suitable rotation transformation, remove the xy-term from

$$11x^2 - 6xy + 3y^2 - 18 = 0 .$$

Describe the conic and sketch it in a single-plane representation. Also find the equation of the reflection of the original conic in the original y-axis and sketch this image in the same Cartesian plane.

POWERS, ROOTS, AND RECIPROCALS 1—100

n	n^2	n^3	$\sqrt{n}$	$\sqrt[3]{n}$	$1/n$
1	1	1	1.000	1.000	1.0000
2	4	8	1.414	1.260	.5000
3	9	27	1.732	1.442	.3333
4	16	64	2.000	1.587	.2500
5	25	125	2.236	1.710	.2000
6	36	216	2.449	1.817	.1667
7	49	343	2.646	1.913	.1429
8	64	512	2.828	2.000	.1250
9	81	729	3.000	2.080	.1111
10	100	1,000	3.162	2.154	.1000
11	121	1,331	3.317	2.224	.0909
12	144	1,728	3.464	2.289	.0833
13	169	2,197	3.606	2.351	.0769
14	196	2,744	3.742	2.410	.0714
15	225	3,375	3.873	2.466	.0667
16	256	4,096	4.000	2.520	.0625
17	289	4,913	4.123	2.571	.0588
18	324	5,832	4.243	2.621	.0556
19	361	6,859	4.359	2.668	.0526
20	400	8,000	4.472	2.714	.0500
21	441	9,261	4.583	2.759	.0476
22	484	10,648	4.690	2.802	.0455
23	529	12,167	4.796	2.844	.0435
24	576	13,824	4.899	2.884	.0417
25	625	15,625	5.000	2.924	.0400
26	676	17,576	5.099	2.962	.0385
27	729	19,683	5.196	3.000	.0370
28	784	21,952	5.292	3.037	.0357
29	841	24,389	5.385	3.072	.0345
30	900	27,000	5.477	3.107	.0333
31	961	29,791	5.568	3.141	.0323
32	1,024	32,768	5.657	3.175	.0312
33	1,089	35,937	5.745	3.208	.0303
34	1,156	39,304	5.831	3.240	.0294
35	1,225	42,875	5.916	3.271	.0286
36	1,296	46,656	6.000	3.302	.0278
37	1,369	50,653	6.083	3.332	.0270
38	1,444	54,872	6.164	3.362	.0263
39	1,521	59,319	6.245	3.391	.0256
40	1,600	64,000	6.325	3,420	.0250
41	1,681	68,921	6.403	3.448	.0244
42	1,764	74,088	6.481	3.476	.0238
43	1,849	79,507	6.557	3.503	.0233
44	1,936	85,184	6.633	3.530	.0227
45	2,025	91,125	6.708	3.557	.0222
46	2,116	97,336	6.782	3.583	.0217
47	2,209	103,823	6.856	3.609	.0213
48	2,304	110,592	6.928	3.634	.0208
49	2,401	117,649	7.000	3.659	.0204
50	2,500	125,000	7.071	3.684	.0200
51	2,601	132,651	7.141	3.708	.0196
52	2,704	140,608	7.211	3.733	.0192
53	2,809	148,877	7.280	3.756	.0189
54	2,916	157,464	7.348	3.780	.0185
55	3,025	166,375	7.416	3.803	.0182
56	3,136	175,616	7.483	3.826	.0179
57	3,249	185,193	7.550	3.849	.0175
58	3,364	195,112	7.616	3.871	.0172
59	3,481	205,379	7.681	3.893	.0169
60	3,600	216,000	7.746	3.915	.0167
61	3,721	226,981	7.810	3.936	.0164
62	3,844	238,328	7.874	3.958	.0161
63	3,969	250,047	7.937	3.979	.0159
64	4,096	262,144	8.000	4.000	.0156
65	4,225	274,625	8.062	4.021	.0154
66	4,356	287,496	8.124	4.041	.0152
67	4,489	300,763	8.185	4.062	.0149
68	4,624	314,432	8.246	4.082	.0147
69	4,761	328,509	8.307	4.102	.0145
70	4,900	343,000	8.367	4.121	.0143
71	5,041	357,911	8.426	4.141	.0141
72	5,184	373,248	8.485	4.160	.0139
73	5,329	389,017	8.544	4.179	.0137
74	5,476	405,224	8.602	4.198	.0135
75	5,625	421,875	8.660	4.217	.0133
76	5,776	438,976	8.718	4.236	.0132
77	5,929	456,533	8.775	4.254	.0130
78	6,084	474,552	8.832	4.273	.0128
79	6,241	493,039	8.888	4.291	.0127
80	6,400	512,000	8.944	4.309	.0125
81	6,561	531,441	9.000	4.327	.0123
82	6,724	551,368	9.055	4.344	.0122
83	6,889	571,787	9.110	4.362	.0120
84	7,056	592,704	9.165	4.380	.0119
85	7,225	614,125	9.220	4.397	.0118
86	7,396	636,056	9.274	4.414	.0116
87	7,569	658,503	9.327	4.431	.0115
88	7,744	681,472	9.381	4.448	.0114
89	7,921	704,969	9.434	4.465	.0112
90	8,100	729,000	9.487	4.481	.0111
91	8,281	753,571	9.539	4.498	.0110
92	8,464	778,688	9.592	4.514	.0109
93	8,649	804,357	9.644	4.531	.0108
94	8,836	830,584	9.695	4.547	.0106
95	9,025	857,375	9.747	4.563	.0105
96	9,216	884,736	9.798	4.579	.0104
97	9,409	912,673	9.849	4.595	.0103
98	9,604	941,192	9.899	4.610	.0102
99	9,801	970,299	9.950	4.626	.0101
100	10,000	1,000,000	10.000	4.642	.0100

ANSWERS

Chapter 1

EXERCISE 1.1 (page 2)

	Declarative	Statement	Open sentence		Declarative	Statement	Open sentence
1.	Yes	False	No	**14.**	Yes	True	No
2.	No	No	No	**15.**	Yes	No	Yes
3.	Yes	No	Yes	**16.**	No	No	No
4.	Yes	True	No	**17.**	No	No	No
5.	Yes	True	No	**18.**	Yes	True	No
6.	Yes	True	No	**19.**	Yes	No	Yes
7.	Yes	False	No	**20.**	Yes	False	No
8.	Yes	True	No	**21.**	Yes	False	No
9.	Yes	False	No	**22.**	Yes	False	No
10.	Yes	False	No	**23.**	Yes	False	No
11.	No	No	No	**24.**	Yes	False	No
12.	Yes	No	Yes	**25.**	Yes	No	Yes
13.	Yes	True	No				

EXERCISE 1.2 (page 6)

3. (a) Set, point, line, on, between. (b) Line segment, joining, point, on, circle. (c) Person, citizen, Canada. (d) Number, *Re*. (e) Polygon, sides.

EXERCISE 1.3 (page 7)

	Conjunction	Disjunction		Conjunction	Disjunction
1.	True	True	**7.**	False	True
2.	True	True	**8.**	True	True
3.	False	True	**9.**	False	True
4.	False	False	**10.**	False	True
5.	False	False	**11.**	False	False
6.	True	True	**12.**	True	True

EXERCISE 1.4 (page 9)

1.	$4 + 5 \neq 20$.	The true one is negation.
2.	Not all right angles are equal.	Statement.
3.	The world will not end on Jan. 16, 2176.	Truth unknowable.
4.	The sum of two consecutive integers is not always an even integer.	Negation.

5.	-3 is not a positive integer.	Negation.
6.	π is not a rational number.	Negation.
7.	$3 + 1 \leq 2$.	Statement.
8.	$x^2 + y^2 = 16$ is not the equation of a line.	Negation.
9.	$\sqrt{2}$ is a rational number.	Statement.
10.	Toronto is not the capital city of Canada.	Negation.
11.	$2 + 3 \neq 5$ or not all dogs have four legs.	Negation.
12.	Not all right angles are equal or the graph of $y = 3x^2$ is not a circle.	Negation.
13.	$2 + 3 \neq 5$ and dogs do not have four legs.	Statement.
14.	Not all right angles are equal and the graph of $y = 3x^2$ is not a circle.	Statement.
15.	Not all similar triangles are congruent and -3 is not a natural number.	Negation.
16.	100 is not 20% of 800 or the angle in a semicircle is not a right angle.	Statement
17.	π is not a rational number and $\sqrt{2}$ is not a rational number.	Negation.
18.	Alexander the Great was not a Roman citizen or $2x + 1 \neq 3$.	Negation.

EXERCISE 1.5 (page 11)

		$p \Rightarrow q$	$q \Rightarrow p$		$p \Rightarrow q$	$q \Rightarrow p$
1.	(a)	True	True	(e)	True	True
	(b)	True	False	(f)	True	True
	(c)	True	True	(g)	False	True
	(d)	True	True	(h)	False	True

3. (a) Statement. (b) Statement. (c) Statement. (d) Statement.
(e) Statement. (f) Negation. (g) Statement. (h) Statement.

4. Yes. **5.** Yes. **6.** No.

EXERCISE 1.7 (page 17)

1. (a) For all x. (b) For some x. (c) For some x. (d) For some x.
(e) For all parallelograms. (f) All triangles are not parallelograms.
(g) Some triangles are not isosceles. (h) For all x.
(i) For all cyclic quadrilaterals. (j) Some rectangles are not squares.

2. (a) For some $x \in Re$, $|x| \neq x$. (b) For all $x \in N$, $x^2 \neq 2$.
(c) Some rational numbers are not real numbers.
(d) For some $n \in N$, $n^2 - n + 41$ is not a prime integer.
(e) Some triangles are isosceles. (f) All triangles are isosceles.
(g) All students are clever. (h) Some students are clever.
(i) Some students are not clever.
(j) For all numbers x, y, $x^2 - y^2 \neq (x - y)^2$.
(k) For all $x \in Re$, $x^2 \neq -4$.
(l) For some positive integer x, $x + 4 \geq 10$.

EXERCISE 1.8 (page 22)

1. Substitution. **2.** Detachment. **3.** Invalid.
4. Disjunction. **5.** Invalid. **6.** Contrapositive.
7. Syllogism. **8.** Invalid. **9.** Invalid.
10. Contrapositive.

EXERCISE 1.10 (page 28)

1. (a) $p \Rightarrow q$. (b) $r \Rightarrow q$. (c) $q \wedge r$. (d) $p \vee r$.
(e) $(r \wedge q) \Rightarrow p$. (f) $\sim r \wedge q$. (g) $\sim r \Rightarrow \sim p$.
(h) $r \Rightarrow (\sim q \vee \sim p)$. (i) $q \Leftrightarrow (\sim r \wedge p)$.

3. (a) $\sim p \vee q$. (b) $p \wedge (\sim q \vee \sim r)$.
(c) $\exists_x, x \in N(2x \text{ is odd})$; $\sim \forall_x, x \in N(2x \text{ is even})$.
(d) $p \wedge \sim q$; $\sim(p \Rightarrow q)$.
(e) $\sim \exists_x, x \in Re\,(2x^2 + 7 = 3)$; $\forall_x, x \in Re, (2x^2 + 7 \neq 3)$.
(f) $\sim \forall_x\,(P_x \Rightarrow Q_x)$; $\exists_x(P_x \Rightarrow Q_x))$; $\exists_x(P_x \wedge \sim Q_x)$.
(g) $\sim \exists_x(S_x \Rightarrow P_x)$; $\forall_x(\sim(S_x \Rightarrow P_x))$; $\forall_x(S_x \wedge \sim P_x)$.

4. (a) All even integers are divisible by 2.
(b) There exists an integer both prime and even.
(c) There exists no integer both prime and even.
(d) Every integer is composite or divisible by 2.
(e) If an integer is prime, it is not divisible by 2.
(f) There exists an integer that is prime if and only if it is even.

REVIEW EXERCISE 1 (page 29)

4. (a) Syllogism. (b) Disjunction. (c) Invalid. (d) Syllogism.
(e) Contrapositive. (f) Invalid. (g) Substitution.
(h) Invalid. (i) Equivalence. (j) Invalid.

7. (a) True. (b) False. (c) True.

9. $\forall_\epsilon, \epsilon > 0(\exists_\delta, \delta > 0(|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon))$.

Chapter 2

EXERCISE 2.1 (page 35)

	Set	Domain	Range
1. (a)	$\{(3, 2), (3, 1)\ (3, 0)\}$	$\{x \mid x = 3\}$	$\{y \mid y \in I, 0 \leq y \leq 2\}$.
(b)	$\{(4, 5), (5, 6), (6, 7), (7, 8)\}$	$\{x \mid x \in I, 4 \leq x \leq 7\}$	$\{y \mid y \in I, 5 \leq y \leq 8\}$.
(c)	$\{(x, \mathrm{y}) \mid x, y \in N \neq x + y < 6\}$	$\{x \mid x \in N, x < 6\}$	$\{y \mid y \in N, y < 6\}$.
(d)	$\{(x, y) \mid x, y \in I, xy = 4\}$	$\{x \mid x \in I, \|x\| \leq 4\}$	$\{y \mid y \in I, \|y\| \leq 4\}$.

2. (a) $(1, \frac{1}{2}), (0, -1), (-4, 2), (1, -5)$. (b) $(\sqrt{2}, \sqrt{\pi}), (-3, 0), (-\frac{1}{3}, \frac{1}{5})$.

3. $M = \{-2, 0, 2\}$, $N = \{3, 5, 7, 9\}$

$$M \times N = \left\{\begin{array}{l}(-2, 3), (-2, 5), (-2, 7), (-2, 9), \\ (0, 3), (0, 5), (0, 7), (0, 9), \\ (2, 3), (2, 5), (2, 7), (2, 9).\end{array}\right\}$$

4. $M \cap N = \phi$, the null set
$M \cup N = \{-2, 0, 3, 5, 7, 9, 2\}$.

5. $P = \{(-2, 3), (0, 5), (2, 7)\}$
$D_P = \{-2, 0, 2\}$
$R_P = \{3, 5, 7\}$.

6. (a) $\{x \mid x \in I, x \neq 0\}$. (b) $\{x \mid x \in I, x \neq -1\}$. (c) $\{x \mid x \in I, |x| \neq 2\}$.
(d) $\{x \mid x \in I\}$. (e) $\{x \mid \sqrt{x} \in I\}$. (f) $\{x \mid x \in I\}$.
(g) $\{x \mid x \in I, x \neq -3, x \neq 2\}$. (h) $\{x \mid x \in I\}$.
(i) $\{x \mid x \in I\}$. (j) $\{x \mid x \in I, x < 0, \sqrt{-x} \in I\}$.

7. (a) $(1, 3), (3, 1), (0, 2), (2, 0)$. (b) $(0, 1), (1, 4), (2, 7), (3, 10)$.
(c) $(1, 0), (2, 0), (3, 0), (4, 0)$. (d) $(0, 0), (1, 1), (2, 2), (3, 3)$.
(e) $(4, 16), (3.5, 12.25), (3.25, 10.5625), (4.5, 20.25)$.
(f) $(2, 2), (1, 1), (-1, 1), (-4, 4)$.

EXERCISE 2.2 (page 38)

1. (i) $x > 0, y > 0$. (ii) $x < 0, y > 0$. (iii) $x < 0, y < 0$. (iv) $x > 0, y < 0$.

2.

		x-intercept	y-intercept
(a)	$x + y = 1$	$x = 1$	$y = 1$.
(b)	$2x + y = 5$	$x = 2.5$	$y = 5$.
(c)	$y = x^2$	$x = 0$	$y = 0$.
(d)	$y - 1 = 3(x - 2)^2$	---	$y = 13$.
(e)	$x^2 + y^2 = 30$	$x = \pm\sqrt{30}$	$y = \pm\sqrt{30}$.
(f)	$xy = 0$	$x \in Re$	$y \in Re$.

3. (a) A straight line parallel to the x-axis with $y = 4$.
(b) A straight line parallel to the y-axis with $x = -2$.
(c) The x-axis. (d) The y-axis.
(e) The first and second quadrants. (f) The first and fourth quadrants.
(g) The first quadrant. (h) The third quadrant.

4. (a) $\{(-4, 16), (-2, 4), (0, 0), (2, 4), (4, 16)\}$.
(b) $\{(-4, 4), (-2, 2), (0, 0), (2, 2), (4, 4)\}$.
(c) $\{(-4, -7), (-2, -3), (0, 1), (2, 5), (4, 9)\}$.
(d) $\{(-4, 4), (-2, 2), (0, 0), (2, -2), (4, -4)\}$.

EXERCISE 2.3 (page 42)

1. (a) $g(1) = -3$. (b) $g(0) = -5$. (c) $g(1) = -7$.
(d) $g(\sqrt{2}) = 2\sqrt{2} - 5$. (e) $g(13) = 21$. (f) $g(-40) = -85$.
(g) $g(2x) = 4x - 5$. (h) $g(x + 1) = 2x - 3$. (i) $g[g(x) = 4x - 15$.
(j) $g(k) = 2k - 5$. (k) $g(x + a) = 2x + 2a - 5$. (l) $g(\pi^2) = 2\pi^2 - 5$.

2. (a) 1. (b) -1. (c) -8. (d) $+1$. (e) 1. (f) 1. (g) 9. (h) -1.

3. The left and middle mappings are functions; the right mapping is not. From left to right, the sets of ordered pairs are $\{(6, -1), (5, -1), (4, -2), (3, -2), (2, -6), (1, -6), (0, -6)\}$, $\{(12, 0), (10, -2), (8, -4), (6, -6), (4, -8), (2, -10), (0, -12)\}$, and $\{(9, 7), (8, 7), (7, 7), (6, 7), (5, 6), (5, 4), (4, 4), (3, 4), (3, 3)\}$.

4. (a) is a function but (b) is not.

6. (a) $\{-6, -3, 0, 3, 6\}$. (b) $\{7, 6, 5, 4, 3\}$. (c) $\{-\frac{3}{2}, -3, 3, \frac{3}{2}\}$.
(d) $\{0, -3, -4\}$.

8. (a) $\{x \mid -8 \leq x \leq 27\}$. (b) $\{x \mid 0 \leq x \leq 3\}$.

9. $\{x \mid f(x) = g(x)\} = \{3, -2\}$.

10. (a) $(0, 17), (1, 8), (2, 5), (3, 8), (4, 17)$.
(b) $(1, 8), (2, 4), (3, \frac{8}{3}), (4, 2), (8, 1)$.

(c) $(0,4)$, $(1,4)$, $(-1,3)$, $(-2,4)$, $(3,7)$.
(d) $(1,-2)$, $(2,-\frac{3}{2})$, $(5,0)$, $(9,2)$, $(11,3)$.

13. (a), (b), (c). **14.** $\{x \mid 1 \leq x \leq 2\}$. **15.** $\{x \mid 0 \leq x \leq 18\}$.

16. $\{\ldots, -14, -7, 0, 7, 14, \ldots\}$

17. (a) 0, 1. (b) none. (c) 0. (d) the integers.

EXERCISE 2.4 (page 48)

1. (i) $\{(-2,-8), (-1,-3), (0,2), (1,7), (2,12)\}$.
(ii) $(-2,-8), (-1,-1), (0,0), (1,1), (2,8)\}$.
(iii) $(-2,2), (-1,1), (0,0), (1,1), (2,2)\}$.

2. (i) $(-8,-2), (-3,-1), (2,0), (7,1), (12,2)\}$ Yes.
(ii) $\{(-8,2), (-1,-1), (0,0), (1,1), (8,2)\}$ Yes.
(iii) $\{(2,-2), (1,-1), (0,0), (1,1), (2,2)\}$ No.

3. Pairs in A^{-1} $\{(3,4), (3,5), (3,6)\}$ A^{-1} is not a function.

4. (a) $\{(x, y) \mid 3y = 2x - 1\}$. (b) $\{(x, y) \mid y = x^{1/3}\}$.
(c) $\{(a, b) \mid b = a + 3\}$. (d) $\{(s, t) \mid t^2 - s^2 = 16\}$.

5. (a) 9. (b) 7. (c) -4. (d) 6. (e) 2.5. (f) -5.

6. The defining sentence for g^{-1} is $g^{-1} = \{(x, y) \mid 2y = x + 5\}$.
g is a function. g^{-1} is a function.
(a) 6. (b) 13. (c) $\dfrac{11}{+4}$. (d) 9. (e) -5. (f) 2.5.

7. The inverse of f is f. **8.** (a) f. (b) ϕ.

9. (b) and (d). **10.** Yes. **11.** No. **12.** Yes.

13. (b) Domain $\{x \mid x \in Re,\ x \neq -2\}$ Range $\{y \mid y \in Re,\ y \neq 0\}$.
(c) Domain $\{x \mid x \in Re,\ x \neq 0\}$ Range $\{y \mid y \in Re,\ y \neq -2\}$.
(d) Yes. **14.** (a) $\{1, -1\}$, $\{0\}$, ϕ. (b) $\{3\}$, ϕ.

15. (a) $f^{-1}(\{x \mid 0 < x < 1\}) = \{x \mid 0 < x < 1\}$.
(b) $f^{-1}(\{x \mid -1.5 < x < 3.7\}) = \{x \mid -2 < x < 4\}$.

EXERCISE 2.5 (page 53)

1. (a) $\{(6,2), (4,3), (1,4), (0,5)\}$. (b) $\{(1,0), (0,4)\}$.
(c) $\{(3,-4), (5,4), (-6,0)\}$.

2. (a) $f(x) \in Re$. (b) $f^{-1}(x) = x + \frac{4}{5}$. (c) $f^{-1}(x) \in Re$ for $x \in Re$.

3. (a) $f(x) \in Re$. (b) $f^{-1}(x) = 2x + \frac{5}{3}$. (c) $f^{-1}(x) \in Re$ for $x \in Re$.

4. (a) $f(x) \in Re, f(x) \geq 0$. (b) $f^{-1}(x) = \sqrt{x^2 + 16}$.
(c) $f^{-1}(x) \in Re$ for $x \in Re$.

5. (a) $f(x) \in Re$. (b) $f^{-1}(x) = 1 - \dfrac{1}{x}$.
(c) $f^{-1}(x) \in Re$ for $x \in Re,\ x \neq 0$; $f^{-1}(x) \neq 1$.

6. (a) $f(x) \in Re, f(x) \geq 0$. (b) $f^{-1}(x) = 3 \pm x$. (c) $f^{-1}(x) \in Re$ for $x \in Re$.

7. (a) $f(x) \in Re^+$. (b) $f^{-1}(x) = \sqrt{x} + 3$
(c) for $x \in Re$, $x > -3$, $f^{-1}(x) \in Re^+$.

8. Range $\{f(x) \mid f(x) \in Re, f(x) \geq -4\}$. **9.** Range $\{g(x) \mid g(x) \in Re, g(x) \leq 27\}$.

10. Range $\{h(x) \mid 0 \leq h(x) \leq 3\}$. **11.** (8) Yes. (9) No. (10) No.

12. (2) $1 - 1$. (3) $1 - 1$. (4) Not $1 - 1$. (5) $1 - 1$. (6) Not $1 - 1$.
(7) Not $1 - 1$. (8) $1 - 1$. (9) Not $1 - 1$. (10) Not $1 - 1$.

13. Identical. **15.** Both functions are identical with their inverses.

16. (a) $\{(x, y) \mid x^2 + y^2 = 25\}$ (b) $\{(x, y) \mid xy = 8\}$.

REVIEW EXERCISE 2 (page 54)

2. $A \cap B = \{3\}$ $A \cup B = \{1, 2, 3, 4, 5, 7\}$.

3. (a) -3. (b) -3. (c) $-\frac{9}{2}$. (d) $||x| - 5| - 5$. (e) $x^2 - 5$.
(f) $|a - b| - 5 = b - a - 5$.

4. (a) $\{x \mid x \in Re\}$ $\{f(x) \mid f(x) \in Re, f(x) \geq -2\}$.
(b) $\{x \mid x \in Re, x \neq 0\}$ $\{f(x) \mid f(x) \in Re\}$.
(c) $\{x \mid x \in Re^+\}$ $\{f(x) \mid f(x) \in Re^+\}$.
(d) $\{x \mid x \in Re\}$ $\{f(x) \mid f(x) \in Re, f(x) \geq 1\}$.
(e) $\{x \mid x \in Re\}$ $\{f(x) \mid f(x) \leq 0\}$.
(f) $\{x \mid x \in Re\}$ $\{f(x) \mid 0 \leq f(x) \leq \frac{1}{5}\}$.

8. (a) 2. (b) 2. (c) 5. (d) $\sqrt[3]{16}$. (e) 1. (f) 5.

9. $B^{-1} = \{(3, 2), (4, 7), (5, 8)\}$. B^{-1} is a function.

10. (a) $|y| \leq 6$. (b) $7 \leq y \leq 119$. **11.** (a) Yes. (b) No. **13.** 8, -6.

16. Domain of $t : x \in Re$. Range of $t : y \in Re, y \geq -11$. $t^{-1} : x \to t + \sqrt{11 - x}$.
Domain $\{x \mid x \in Re, x \leq 11\}$. Range of $t : y \in Re, y \geq 6$.

17. (b) $\{x \mid x \in Re\}$ $\{f \mid f \in Re, 0 < f \leq 1\}$.
(d) $\{x \mid x \in Re, x \neq 0\}$ $\{f \mid f \in Re\}$. (e) No.

19. The second inclusion is proper for $f : Re \to Re, f(x) = x^2$,
$A = \{x \mid 10 \leq x \leq 3\}$, $B = \{x \mid 1 \leq x \leq 5\}$.

20. The first inclusion is proper for $f : Re \to Re$, $f(x) = x^2$, $D = \{x \mid 0 \leq x \leq 1\}$.
The second inclusion is proper for $f : Re \to Re$, $f(x) = x^2$, $H = \{1, -1\}$.

Chapter 3

EXERCISE 3.1 (page 62)

1. (a) $x^2 + y^2 = 16$. (b) $25x^2 + 25y^2 = 9$. (c) $x^2 + y^2 = 8$.
(d) $3x^2 + 3y^2 = 1$. (e) $16x^2 + 16y^2 = 7$.

2. (a) $x^2 + y^2 = 169$. (b) $x^2 + y^2 = 5$. (c) $x^2 + y^2 = 27$.
(d) $x^2 + y^2 = 29$. (e) $x^2 + y^2 = 41$. (f) $x^2 + y^2 = 100$.

3. (a) 7. (b) $2\sqrt{3}$. (c) $\frac{9}{2}$. (d) $\frac{352}{5}$.

4. (a) $(0, \pm 2)$; $(\pm 2, 0)$; $-2 \leq x \leq 2$; $-2 \leq y \leq 2$.
(b) $(0, \pm\sqrt{\frac{3}{3}})$; $(\pm\sqrt{\frac{3}{3}})$; $-\frac{1}{3}\sqrt{3} \leq x \leq \frac{1}{3}\sqrt{3}$; $-\frac{1}{3}\sqrt{3} \leq y \leq \frac{1}{3}\sqrt{3}$.
(c) $(0, \pm\frac{8}{5})$; $(\pm\frac{8}{5}, 0)$; $-\frac{8}{5} \leq x \leq \frac{8}{5}$; $-\frac{8}{5} \leq y \leq \frac{8}{5}$.
(d) $(0, \pm\sqrt{6})$; $(\pm\sqrt{6}, 0)$; $-\sqrt{6} \leq x \leq \sqrt{6}$; $-\sqrt{6} \leq y \leq \sqrt{6}$.

5. (a) x-axis, y-axis, origin. (b) y-axis. (c) Origin.
(d) None. (e) x-axis. (f) x-axis, y-axis, origin.
(g) y-axis. (h) Origin.

6. (a) On. (b) Inside. (c) On. (d) On. (e) On. (f) Outside.

	x-intercept	y-intercept	Domain	Range	Symmetry
9. (a)	$(\pm 4, 0)$	None	$\|<\|x \leq 4$	$\|y\| \leq 4$	x-axis, y-axis, origin.
(b)	$\pm\sqrt{5}$	$-1, +5$	$\|x\| \leq 3$	$-1 \leq y \leq 5$	y-axis.
(c)	$-4, 2$	$\pm 2\sqrt{2}$	$-4 \leq x \leq 2$	$\|y\| \leq 3$	x-axis.

EXERCISE 3.2 (page 64)

1. $3\sqrt{2}$. 2. $x^2 + y^2 = 25$.

		Length of cord	Equation of cord
3.	(a)	6	$y = 4$
	(b)	$4\sqrt{5}$	$x + 2y = 5$
	(c)	4	$8x - 10y = -41$
	(d)	No solution.	

4. The point (5, 6) is outside the circle.

5. (a) 6. (b) $\sqrt{71}$. (c) $\sqrt{55}$. (d) $4\sqrt{22}$. (e) No solution.

6. Point (3, 4) is inside the circle. 7. $\sqrt{x_1^2 + y_1^2 - r^2}$; $x_1^2 + y_1^2 - r^2 \geq 0$.

10. $x^2 + y^2 - 4x - 2y = 20$. 11. $2\sqrt{15}$.

EXERCISE 3.3 (page 67)

	x-intercept	y-intercept	Domain	Range	Symmetry
1.	0	0	$x \in Re$	$y \leq 0$	y-axis
2.	0	0	$x \in Re$	$y \geq 0$	y-axis
3.	0	0	$x \in Re$	$y \geq 0$	y-axis
4.	0	0	$x \in Re$	$y \geq 0$	y-axis
5.	0	0	$x \in Re$	$y \leq 0$	y-axis
6.	0	0	$x \in Re$	$y \geq 0$	y-axis

13. $k = 8$. 14. y-axis. 16. $y + 5 = 3(x - 2)^2$. 17. $y - 2 = -4(x + 3)^2$.

18. The graph changes from a straight line for $a = 0$ to a parabola for $a > 0$; as a increases the parabola becomes "narrower." As $a \to \infty$, the graph approaches (but never becomes) that of the upper half of the y-axis.

EXERCISE 3.4 (page 72)

	x-intercepts	y-intercepts	Domain	Range	Symmetry
1.	± 5	± 4	$\|x\| \leq 5$	$\|y\| \leq 4$	x-axis, y-axis, origin.
2.	± 5	$\pm 2\sqrt{3}$	$\|x\| \leq 5$	$\|y\| \leq 2\sqrt{3}$	x-axis, y-axis, origin.
3.	± 3	± 1	$\|x\| \leq 3$	$\|y\| \leq 1$	x-axis, y-axis, origin.
4.	$\pm 2\sqrt{5}$	$\pm 2\sqrt{2}$	$\|x\| \leq 2\sqrt{5}$	$\|y\| \leq 2\sqrt{2}$	x-axis, y-axis, origin.
5.	$\pm\sqrt{3}$	± 1	$\|x\| \leq \sqrt{3}$	$\|y\| \leq 1$	x-axis, y-axis, origin.
6.	± 8	± 3	$\|x\| \leq 8$	$\|y\| \leq 3$	x-axis, y-axis, origin.

7. (a)

	1	2	3	4	5	6
Semi-major	5	5	3	$2\sqrt{5}$	$\sqrt{3}$	8
Semi-minor	4	$2\sqrt{3}$	1	$2\sqrt{2}$	1	3

(b) $(\pm 5, 0)$, $(\pm 5, 0)$, $(\pm 3, 0)$, $(\pm 2\sqrt{5}, 0)$, $(\pm\sqrt{3}, 0)$, $(\pm 8, 0)$
$(0, \pm 4)$, $(0, \pm 2\sqrt{3})$, $(0, \pm 1)$, $(0, \pm 2\sqrt{2})$, $(0, \pm 1)$, $(0, \pm 3)$.

14. $R = 2$. 15. Circle, $x^2 + y^2 = 4$.

16. (a) Inside. (b) Outside. (c) Inside. (d) On.

EXERCISE 3.5 (page 76)

	x-intercepts	y-intercepts	Domain	Range	Symmetry
1.	± 5	---	$\lvert x\rvert \geq 5$	$y \in Re$	x-axis, y-axis, origin.
2.	± 2	---	$\lvert x\rvert \geq 2$	$y \in Re$	x-axis, y-axis, origin.
3.	± 2	---	$\lvert x\rvert \geq 2$	$y \in Re$	x-axis, y-axis, origin.
4.	$\pm 2\sqrt{5}$	---	$\lvert x\rvert \geq 2\sqrt{5}$	$y \in Re$	x-axis, y-axis, origin.
5.	$\pm \frac{1}{2}$	---	$\lvert x\rvert \geq \frac{1}{2}$	$y \in Re$	x-axis, y-axis, origin.
6.	± 5	---	$\lvert x\rvert \geq 5$	$y \in Re$	x-axis, y-axis, origin.

7.

	Semi-transverse	Semi-conjugate	Vertices
(1)	5	4	$(\pm 5, 0)$
(2)	2	5	$(\pm 2, 0)$
(3)	2	$2\sqrt{5}$	$(\pm 2, 0)$
(4)	$2\sqrt{5}$	7	$(\pm 2\sqrt{5}, 0)$
(5)	$\frac{1}{2}$	1	$(\pm \frac{1}{2}, 0)$
(6)	5	5	$(\pm 5, 0)$

8. The transverse and conjugate axes are equal in length.

15. $k = 1$. **16.** Ellipse, hyperbola.

EXERCISE 3.6 (page 80)

11. $xy = \dfrac{a^2}{2}$ is a rotation of the graph of $x^2 - y^2 = 25$ through $\dfrac{\pi}{4}$ radians.

19. First draw the asymptotes and then sketch the hyperbola so that the asymptotes are the limiting cases of the hyperbola.

REVIEW EXERCISE 3 (page 81)

1. (a) $x^2 + y^2 = 100$. (b) $81x^2 + 81y^2 = 16$.
(c) $x^2 + y^2 = 5$. (d) $9x^2 + 9y^2 = 13$.

2. (a) $x^2 + y^2 = 100$. (b) $x^2 + y^2 = 29$. (c) $x^2 + y^2 = 18$.

3. (a) $\frac{5}{2}$ (b) $2\sqrt{7}$. (c) $\sqrt{\frac{35}{6}}$. (d) $\dfrac{\sqrt{2}}{10}$.

4. (a) x-axis, y-axis, origin. (b) x-axis, y-axis, origin.
(c) y-axis. (d) Origin. (e) Origin. (f) y-axis.
(g) y-axis. (h) x-axis, y-axis, origin.

27. $k = 3$. **28.** $k = 16$. **29.** $k = 4$. **30.** $k = \frac{18}{5}$.

32. (a) $5 < m < 30$. (b) $m < 5$ or $m > 30$.

Chapter 4

EXERCISE 4.1 (page 86)

1. $(0, 2)$ $y = -2, \frac{1}{2}$. **2.** $(0, \frac{3}{5})$ $y = -\frac{3}{5}, \frac{5}{3}$. **3.** $(0, -\frac{15}{16})$ $y = \frac{15}{16}, -\frac{16}{5}$.

4. $(0, -.01)$ $y = .01, -100$. **5.** $x^2 = -16y$. **6.** $x^2 = -20y$.

7. $x^2 = 12y$. **8.** $x^2 - 30y$. **9.** 1. **10.** $-\frac{1}{4}$.

11. -1. **12.** -1. **15.** $(5, 0)$, $(x + 5) = 0, \frac{5}{4}$. **16.** $(-\frac{15}{6}, 0)$. $x = \frac{15}{16}, -\frac{20}{3}$.

17. $(-\frac{1}{32}, 0); x = \frac{1}{32}; -200.$ **18.** $(\frac{5}{8}, 0); x = \frac{5}{8}; 10.$

19. $y^2 = -40x.$ **20.** $y^2 = -\frac{4}{3}x.$ **21.** $y^2 = \frac{28}{5}x.$

22. $y^2 = -\frac{5}{2}x.$ **23.** $-\frac{1}{2}.$ **24.** $-\frac{1}{2}.$

25. $\frac{1}{2}.$ **26.** $-\frac{1}{28}.$ **27.** $(3, 0); y^2 = 12.$

28. $(0, -\frac{12}{7}); x^2 = -\frac{48}{7}y.$ **30.** $y^2 - 4y = 16x - 20.$ **31.** $y^2 = -4px + 4p^2.$

EXERCISE 4.2 (page 96)

1 **1.** $(\pm 3, 0); x = \pm\frac{25}{3}; \frac{3}{5}.$ **2.** $(\pm 3\sqrt{3}, 0); x = \pm 4\sqrt{3}; \frac{1}{2}\sqrt{3}.$

3. $(\pm 2, 0); x = \pm 10; \frac{1}{5}\sqrt{5}.$ **4.** $(\pm\sqrt{2}, 0); x = \frac{5}{2}\sqrt{2}; \frac{1}{5}\sqrt{10}.$

5. $(\pm 2\sqrt{23}, 0); x = \pm\frac{32}{23}\sqrt{23}; \frac{1}{4}\sqrt{23}.$ **6.** $(\pm 5, 0); x = \frac{16}{5}; \frac{5}{4}.$

7. $(\pm\sqrt{10}, 0); x = \pm \dfrac{9}{\sqrt{10}}; \dfrac{\sqrt{10}}{3}.$ **8.** $(\pm 3, 0); x = \pm\frac{5}{3}; \dfrac{3}{\sqrt{5}}.$

9. $\dfrac{x^2}{100} + \dfrac{y^2}{75} = 7.$ **10.** $\dfrac{x^2}{36} + \dfrac{y^2}{32} = 7.$ **11.** $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 7.$

12. $\dfrac{x^2}{49} + \dfrac{16y^2}{343} = 7.$ **13.** $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 7.$ **14.** $\dfrac{x^2}{4} - \dfrac{4y^2}{9} = 7.$

15. $\dfrac{4x^2}{169} - \dfrac{4y^2}{507} = 7.$ **16.** $\dfrac{4x^2}{49} - \dfrac{y^2}{18} = 7.$ **17.** 231.

18. 8. **19.** 3. **20.** $\frac{8}{13}.$ **21.** $\dfrac{x^2}{16} + \dfrac{25y^2}{336} = 1.$

22. $\dfrac{x^2}{2} - \dfrac{y^2}{2} = 7.$ **24.** $\dfrac{(x-1)^2}{16} + \dfrac{(y-1)^2}{15} = 1.$

25. $\dfrac{(x + \frac{15}{4})^2}{\frac{81}{4}} - \dfrac{(y+4)^2}{\frac{405}{16}} = 1.$

EXERCISE 4.3 (page 101)

1. (a) $\dfrac{x^2}{25} + \dfrac{y^2}{9} = 1.$ (b) $\dfrac{x^2}{64} + \dfrac{y^2}{28} = 1.$ (c) $\dfrac{x^2}{49} + \dfrac{y^2}{13} = 1.$

(d) $\dfrac{x^2}{676} + \dfrac{y^2}{144} = 1.$

2. (a) $\dfrac{x^2}{36} - \dfrac{y^2}{9} = 1.$ (b) $\dfrac{x^2}{16} - \dfrac{y^2}{9} - 1.$ (c) $\dfrac{x^2}{64} - \dfrac{y^2}{36} = 1.$

(d) $\dfrac{x^2}{27} - \dfrac{y^2}{9} = 1.$

3. $\dfrac{x^2}{169} + \dfrac{y^2}{25} = 1.$ **4.** $\dfrac{x^2}{49} + \dfrac{y^2}{16} = 1.$ **5.** $\dfrac{3x^2}{25} + \dfrac{2y^2}{35} = 1.$

6. $\dfrac{x^2}{27} + \dfrac{y^2}{18} = 1.$

7. $\dfrac{x^2}{36} - \dfrac{y^2}{64} = 1.$ **8.** $\dfrac{x^2}{14} - \dfrac{y^2}{22} = 1.$ **9.** $\dfrac{x^2}{24} - \dfrac{y^2}{96} = 1.$

10. $\dfrac{x^2}{10} - \dfrac{3y^2}{20} + 1.$

	11.	12.	13.	14.	15.	16.
Semi-major axis	3	4	$4\sqrt{2}$	13	4	$\frac{1}{8}$
Semi-minor axis		2		5		$\frac{1}{12}$
or Conjugate axis	5		$4\sqrt{2}$		$\sqrt{3}$	
Foci	$(\pm\sqrt{34}, 0)$	$(\pm 2\sqrt{3}, 0)$	$(\pm 8, 0)$	$(\pm 12, 0)$	$(\pm\sqrt{19}, 0)$	$\left(\pm\frac{\sqrt{5}}{24}, 0\right)$
Eccentricity	$\sqrt{\frac{34}{3}}$	$\sqrt{\frac{3}{2}}$	$2\sqrt{3}$	$\frac{12}{13}$	$\frac{\sqrt{19}}{4}$	$\frac{\sqrt{5}}{3}$

18. $\frac{x^2}{73} + \frac{16y^2}{73} = 1.$ **19.** $\frac{x^2}{25} + \frac{y^2}{9} = 1.$ **20.** $3x^2 + 4y^2 = 2x + 2y - 2.$

21. $\frac{x^2}{9} - \frac{y^2}{4} = 1.$ **22.** $4x^2 - y^2 = 100.$ **23.** $\frac{x^2}{3025} - \frac{y^2}{86975} = 1.$

25. $\frac{x^2}{5625} - \frac{y^2}{20625} = 1.$

EXERCISE 4.4 (page 107)

1. $(\pm 2, 0)$, $(0, \pm 5)$, $\frac{\sqrt{21}}{5}.$ **2.** $(\pm 1, 0)$ $(0, \pm 3)$, $2\sqrt{2}.$

3. $(\pm\sqrt{3}, 0)$, $(0, \pm 2)$, $\frac{1}{3}\sqrt{3}.$ **4.** $(\pm\frac{1}{9}, 0)$, $(0, \pm\frac{1}{4})$, $\frac{1}{9}\sqrt{65}.$

5. $\frac{x^2}{9} + \frac{y^2}{25} = 1.$ **6.** $\frac{x^2}{288} + \frac{y^2}{324} = 1.$ **7.** $\frac{x^2}{600} + \frac{y^2}{625} = 1.$

8. $\frac{x^2}{289} + \frac{y^2}{64} = 1.$ **9.** $\frac{x^2}{32} + \frac{7y^2}{128} = 1.$ **10.** $\frac{3x^2}{35} + \frac{2y^2}{35} = 1.$

11. $(0, \pm 5)$; $(0, \pm\sqrt{34})$; $\frac{1}{5}\sqrt{34}.$ **12.** $(0, \pm 2)$; $(0, \pm\sqrt{13})$; $\frac{\sqrt{13}}{2}.$

13. $(0, \pm 2)$; $(0, \pm 2\sqrt{2})$; $\sqrt{2}.$ **14.** $(0, \pm\sqrt{3})$; $(0, 2\sqrt{3})$; 2.

15. $\frac{y^2}{9} - \frac{x^2}{16} = 1.$ **16.** $\frac{9y^2}{64} - \frac{9x^2}{80} = 1.$ **17.** $\frac{y^2}{18} - \frac{x^2}{16} = 1.$

18. $\frac{y^2}{12} - \frac{x^2}{12} = 1.$ **19.** $\frac{x^2}{24} - \frac{y^2}{96} = 1.$ **20.** $\frac{517y^2}{993} - \frac{112x^2}{331} = 1.$

25. $\frac{x^2}{12} + \frac{y^2}{16} = 1.$ **26.** $\frac{9y^2}{400} - \frac{9x^2}{500} = 1.$ **29.** 2 feet.

EXERCISE 4.5 (page 111)

1. $(0, 0)$, $(4, 4)$. **2.** $(\frac{1}{2}, 2)$, $(8, -8)$. **3.** $(6, 1)$, $(2, 3)$.

4. $(\pm\sqrt{\frac{45}{23}}, \pm\sqrt{\frac{2}{23}})$. **5.** $(3, -\frac{16}{5})$. **6.** None.

7. $(1, 1)$, $(-3, 5)$. **8.** $(\pm 5.53, \pm 6.67)$. **19.** $(\pm 4, \pm 4)$, 64.

20. $\frac{x^2}{144} + \frac{y^2}{16} = ?.$ **21.** $\frac{1}{2}\sqrt{2}.$ **22.** $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{x}{a}.$

23. (a) $x = \frac{x_1 + x_2}{2}$ $y = \frac{y_1 + y_2}{2}.$ (d) $9y + 32x = 0.$

EXERCISE 4.6 (page 115)

1. $3x - 4y = 5$. **2.** $x\sqrt{2} = y + 4$. **3.** $-8x + y = 10$.
4. $4x - y = 15$. **5.** $y = -\frac{1}{2}x \pm \sqrt{5}$. **6.** $y = -\frac{3}{4}x + 4$.
7. $y = \frac{1}{3}x \pm \frac{11}{3}$. **8.** $y = 2x \pm \frac{1}{2}$.
9. $y = -3x + 6;\ y = \frac{1}{3}x - \frac{10}{3}$. **10.** $y = 5x - 25;\ x + y + 1 = 0$.
11. $y = -1;\ x = 2$. **12.** $y = 2x - 1;\ y = \frac{7}{4} - 1$.
13. $R = \pm 7$. **17.** $(\sqrt{\frac{20}{13}}, -\frac{3}{4}\sqrt{\frac{20}{13}})\ (-\sqrt{\frac{20}{13}}, \frac{3}{4}\sqrt{\frac{20}{13}})$.
21. (a) $(3, 1), (1, -3)$. (b) $y = 2x - 5$. (d) $x_1x + y_1y = r^2$.
22. (a) $k^2 - 8k + 4 = 0$. (b) $k_1 + k_2 = 8,\ k_1k_2 = 4$.
23. (b) $(\frac{2}{3}, -2)$. **26.** (a) $3y + 2x = 0$. (b) $(3, -2), (-3, 2)$.
30. The second theorem implies the first.

EXERCISE 4.7 (page 120)

1. $3y = \pm 2x$. **2.** $x = \pm y$. **3.** $7x = \pm y$. **4.** $x = \pm y$.
5. $\sqrt{14}x = \pm 5y$. **6.** $12x = \pm 5y$. **10.** $x = \pm y$.
13. $xy = -32;\ x = 0,\ y = 0$. **14.** $16x^2 - 25y^2 = 39$.

REVIEW EXERCISE 4 (page 122)

1. $(0, \frac{2}{3});\ y = -\frac{2}{3};\ \pm\frac{4}{3}\sqrt{3}$. **2.** $(\frac{3}{8}, 0);\ x = -\frac{3}{8};\ \frac{8}{3}$.
3. $(0, \frac{15}{16});\ y = \frac{15}{16};\ \pm\frac{1}{2}\sqrt{30}$. **4.** $(-\frac{7}{6}, 0);\ x = \frac{7}{6}, -\frac{6}{7}$.
5. $x^2 = -20y$. **6.** $\frac{x^2}{40} + \frac{y^2}{24} = 1$. **7.** $\frac{x^2}{100} + \frac{y^2}{84} = 1$.
8. $\frac{x^2}{25} + \frac{8y^2}{175} = 1$. **9.** $\frac{x^2}{9} - \frac{y^2}{16} = 1$. **10.** $\frac{x^2}{15} - y^2 = -64$.
11. $\frac{8x^2}{9} + \frac{8y^2}{9} = 1$. **12.** $\frac{x^2}{25} + \frac{y^2}{16} = 1$. **13.** $-\frac{x^2}{25} + \frac{y^2}{200} = 1$.
14. $\frac{y^2}{25} + \frac{64x^2}{975} = 1$. **15.** $\frac{16x^2}{225} + \frac{16y^2}{199} = 1$. **16.** $\frac{x^2}{16} - \frac{3y^2}{20} = 1$.
17. $\frac{x^2}{4} + \frac{y^2}{16} = 1$. **18.** $\frac{x^2}{50} - \frac{y^2}{50} = -1$. **24.** $2x + 3y = 13$.
25. $y = 2x - 2$. **26.** $(8, 8)$. **27.** $(-\frac{5}{4})$.
28. $y = \frac{3}{4}x,\ y = -\frac{5}{4}x - 8$. **29.** $3x^2 = 4000y$.
31. $y = mx - pm^2$. **32.** $y = \frac{1}{2}x - \frac{1}{6}$. **33.** k^2.
34. $y = -\frac{1}{2}x - 2$. **35.** (a) $x = \frac{1}{2}(x_1 + x_2)\ y = \frac{1}{2}(y_1 + y_2)$.
(c) $y = \frac{3}{40}x$.
36. (d) $(-30, 10)$.
37. (a) $5x = \pm 4y$. (b) $6x = \pm y$. (c) $x = \pm y$.
(d) $x\sqrt{5} = \pm y$. (e) $9x = \pm 13y$. (f) $x = 0;\ y = 0$.
38. $9x^2 - 16y^2 = 65$.

Chapter 5

EXERCISE 5.1 (page 132)

1. .74314. **2.** $-.17365$. **3.** .30573. **4.** $-.74314$.
5. $-.17365$. **6.** $-.30573$. **7.** .26795. **8.** .95106.
9. .98481. **10.** .26795. **11.** $-.95106$. **12.** $-.98481$.
13. .29237. **14.** $-.45399$. **15.** $-.70021$. **16.** $-.81915$.
17. .34202. **18.** 2.7475. **19.** $\frac{1}{2}$. **20.** $\sqrt{2}$. **21.** $\sqrt{3}$.
22. $\frac{1}{2}\sqrt{3}$. **23.** $-\frac{1}{2}\sqrt{2}$. **24.** $-\frac{1}{3}\sqrt{3}$. **25.** $-\frac{2}{3}\sqrt{3}$.
26. $-\frac{1}{2}\sqrt{2}$. **27.** $\frac{1}{2}\sqrt{3}$. **28.** -1. **29.** $-\frac{1}{2}\sqrt{3}$.
30. $-\frac{2}{3}\sqrt{3}$. **31.** $\frac{1}{2}\sqrt{2}$. **32.** $\sqrt{3}$. **33.** 2.
34. 27°, 153°. **35.** 32°, 328°. **36.** 47°, 227°. **37.** 47°, 313°.
38. 230°, 310°. **39.** 244°, 296°. **40.** 43°, 317°. **41.** 15°, 165°.
42. 102°, 282°.
45. (a) $-\frac{1}{2}\sqrt{3}$. (b) $-.64279$. (c) $-.36397$. (d) $-.17365$.
(e) $-\frac{1}{3}\sqrt{3}$. (f) 5.7588.

EXERCISE 5.2 (page 138)

2. (a) 1. (b) 0°, 360°. (c) -1. (d) $-180°$, 180°. (e) $-\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$.
4. (a) About origin, and about lines $x = (k + \frac{1}{2})\pi$, $k \in I$.
(b) About lines $x = k\pi$, $k \in I$.
5. (a) 0°, 360°, 720°, 1080°. (b) $-30°$, 330°, 690°, 1050°.
(c) 45°, 405°, 765°, 1125°. (d) $-90°$, 270°, 630°, 900°.
6. (a) 135°, 495°, 855°, 1215°. (b) 60°, 420°, 780°, 1140°.
(c) 150°, 510°, 870°, 1230°. (d) 140°, 500°, 860°, 1220°.
7. (a) $\theta = 45°, 225°$. (b) $\theta = 135°, 315°$. (c) $\theta = 0°, 90°, 360°$.
(d) No value. (e) No value. (f) $\theta = 0°, 270°, 360°$.
8. (a) 1. (b) 1. **9.** (a) 3. (b) k.

EXERCISE 5.3 (page 140)

2. (a) No maximum or minimum. (b) Domain: $\theta \in Re$. Range: $\cot\theta \in Re$.
(c) $\theta = 90° + 360°k$, $k \in I$. (d) 180°. (e) $\theta = 45°, 225°, 405°, 585°$.
7. All values, and the amplitude, are 3 times as great.
8. All values, and the amplitude, are half as great.
9. The period is twice as great.

EXERCISE 5.4 (page 143)

2. The amplitudes in order are .5, 1, 3.
3. (a) 3. (b) $\frac{1}{5}$. (c) 4.3. (d) .002. (e) 2π. (f) 3×10^6.
5. The amplitudes in order are .8, 1, 2.0.
6. (a) 7. (b) .03. (c) 5×10^3. (d) 6×10^{-2}. (e) $\sqrt{2}$. (f) $\frac{2}{5}$.

EXERCISE 5.5 (page 146)

1. (a) 180°. (b) 90°. (c) 720°. (d) 1440°.
2. Periods in order are 120°, 90°. They first repeat after 360°.

3. (a) 360°. (b) 144°. (c) 72°. (d) 900°.
5. (a) Period = 180°. (b) Period = 720°. (c) Period = 2160°.
(d) Period = 1080°. The period is the least common multiple of the individual periods.

EXERCISE 5.6 (page 149)

1. (a) $\frac{\pi}{6}$. (b) $-\frac{3\pi}{4}$. (c) 25°. (d) $-120°$.

2. (a) $1, \frac{\pi}{4}$. (b) $1, -\frac{2\pi}{3}$. (c) $1, -75°$. (d) $1, 150°$.

3. (a) $-\frac{\pi}{3}$. (b) $\frac{3\pi}{4}$. (c) 144°. (d) $\frac{\pi}{\sqrt{2}}$. (e) $-\sqrt{2}$. (f) $-\frac{\pi}{4}$.

4. (a) $\theta = -\frac{2\pi}{3}, -\frac{5\pi}{3}, -\frac{8\pi}{3}, -\frac{11\pi}{3}, \frac{\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$.
(b) $\theta = \frac{3\pi}{4}, -\frac{\pi}{4}, -\frac{5\pi}{4}, -\frac{9\pi}{4}, -\frac{13\pi}{4}, \frac{7\pi}{4}, \frac{11\pi}{4}, \frac{15\pi}{4}$.
(c) $\theta = \frac{\pi}{3}, -\frac{2\pi}{3}, -\frac{5\pi}{3}, -\frac{8\pi}{3}, -\frac{11\pi}{3}, \frac{4\pi}{3}, \frac{7\pi}{3}, \frac{10\pi}{3}$.
(d) $\theta = 63°, 243°, 423°, 603°, -117°, -297°, -477°, -657°$.
(e) $\theta = 50°, -130°, -310°, -490°, -670°, 230°, 410°, 590°$.
(f) $\theta = 60°, -120°, -300°, -480°, -660°, 240°, 420°, 600°$.

5.

	Maxima	Minima
(a)	$\theta = 45°, 405°, 765°$.	$\theta = 225°, 585°, 945°$.
(b)	$\theta = \frac{\pi}{3}, \frac{7\pi}{3}, \frac{13\pi}{3}$.	$\theta = \frac{4\pi}{3}, \frac{10\pi}{3}, \frac{16\pi}{3}$.
(c)	$\theta = \frac{5\pi}{4}, \frac{13\pi}{4}, \frac{21\pi}{4}$.	$\theta = \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}$.
(d)	$\theta = \frac{5\pi}{3}, \frac{11\pi}{3}, \frac{17\pi}{3}$.	$\theta = \frac{\pi}{6}, \frac{13\pi}{6}, \frac{25\pi}{6}$.
(e)	$\theta = 125°, 485°, 845°$.	$\theta = 305°, 665°, 1025°$.
(f)	$\theta = \frac{7\pi}{6}, \frac{19\pi}{6}, \frac{31\pi}{6}$.	$\theta = \frac{3\pi}{3}, \frac{5\pi}{3}, \frac{8\pi}{3}$.

EXERCISE 5.7 (page 152)

1.

	Amplitude	Period	Phase Angle	Shift
(a)	.5	2π	0	0
(b)	.5	2π	$\frac{\pi}{4}$	$-\frac{\pi}{4}$
(c)	1	2π	$\frac{\pi}{4}$	$-\frac{\pi}{4}$
(d)	.5	$\frac{2\pi}{3}$	0	0
(e)	1	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$-\frac{\pi}{4}$
(f)	.5	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$-\frac{\pi}{4}$

2.

	Amplitude	Period	Phase Angle	Shift
(a)	1	2π	0	0
(b)	2	2π	$-\frac{2\pi}{3}$	$\frac{2\pi}{3}$
(c)	2	2π	0	0
(d)	2	2π	$-\frac{2\pi}{3}$	$\frac{2\pi}{3}$
(e)	1	$\frac{\pi}{2}$	0	0
(f)	2	$\frac{\pi}{2}$	$-\frac{2\pi}{3}$	$\frac{2\pi}{3}$

3. Phase angle = $-\pi$; shift = π. **4.** Phase angle = π; shift = $-\frac{\pi}{3}$.

REVIEW EXERCISE 5 (page 154)

1. .77715. **2.** .66913. **3.** .24933. **4.** −.77715.
5. −.66913. **6.** −.24933. **7.** .32492. **8.** .96593.
9. .58779. **10.** .32492. **11.** −.96593. **12.** −.58779.
13. $\frac{\sqrt{3}}{2}$. **14.** $\frac{\sqrt{3}}{2}$. **15.** −1. **16.** −1.
17. $\frac{\sqrt{2}}{2}$. **18.** $\sqrt{\frac{1 + \cos 45^\circ}{2}} = .92388$. **19.** $1 + \sqrt{2} = 2.4142$.
20. 2. **21.** .98481. **23.** The amplitudes are: (i) 1. (ii) 3 (iii) $\frac{1}{3}$.
24. (i) 2π. (ii) 2π. (iii) 2π.

25.

	Amplitude	Phase Angle
(a)	1	$\frac{\pi}{3}$
(b)	3	$-\frac{\pi}{2}$
(c)	2.5	50°
(d)	.375	-35°
(e)	7.5	$\frac{2\pi}{3}$
(f)	4	-135°.

26.

	Amplitude	Period
(a)	1	$\frac{2\pi}{3}$
(b)	2	4π
(c)	3	π
(d)	.25	$\frac{8\pi}{3}$
(e)	1	24π
(f)	7.5	$\frac{8\pi}{25}$.

27.

	Amplitude	Phase Angle
(a)	2	$-\frac{\pi}{3}$
(b)	1.25	30°
(c)	3	$\frac{5\pi}{6}$
(d)	1	-120°
(e)	3	$-\frac{\pi}{2}$
(f)	1.5	$-\frac{3\pi}{4}$.

28.

	Amplitude	Period	Phase Angle	Shift
(a)	2.5	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$-\frac{\pi}{4}$
(b)	$\frac{1}{5}$	π	$-\frac{\pi}{3}$	$\frac{\pi}{6}$
(c)	3	π	90°	-45°
(d)	3.5	4π	$-12\frac{1}{2}^\circ$	25°
(e)	4	$\frac{2\pi}{3}$	45°	-15°
(f)	2	$\frac{2\pi}{3}$	-90°	30°.

32.

	Minimum	θ	Maximum	θ
(a)	0	$\frac{3\pi}{2} + 2n\pi$	2	$\frac{\pi}{2} + 2n\pi$
(b)	−1	$\frac{\pi}{10} + \frac{2n\pi}{5}$	5	$\frac{3\pi}{10} + \frac{2n\pi}{5}$
(c)	$\frac{1}{2}$	$\frac{3\pi}{2} + 2n\pi$	2	$\frac{\pi}{2} + 2n\pi$
(d)	.5403	$n\pi$	1	$\frac{(2n+1)\pi}{2}$
(e)	8.0806	$\frac{(2n+1)\pi}{2}$	9	$n\pi$

Chapter 6

EXERCISE 6.1 (page 161)

1. $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$. 2. $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$. 3. $-\frac{1}{2}$. 4. $2-\sqrt{3}$.
5. $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$. 6. $\frac{4\sqrt{2}-3\sqrt{3}}{5}$. 7. $\frac{1}{39}(10+12\sqrt{5})$.
8. $\frac{1}{39}(5\sqrt{5}-24)$. 9. $\frac{1740+338\sqrt{5}}{101}$. 10. $\frac{870-169\sqrt{5}}{310}$.
11. $\frac{1}{39}(10-12\sqrt{5})$. 12. $\frac{1}{39}(5\sqrt{5}+24)$. 13. $\frac{-4}{(39)^2}(30+119\sqrt{5})$.
14. $\frac{480\sqrt{5}-119}{1521}$. 15. $\frac{30+119\sqrt{5}}{288}$. 29. $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$.
30. $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$. 31. $\sqrt{3}-2$. 32. $\frac{\sqrt{2}}{4}(\sqrt{3}+1)$.
33. $\frac{\sqrt{2}}{4}(\sqrt{3}-1)$. 34. $-\sqrt{3}$. 35. $\frac{\sqrt{3}}{2}$. 36. $\frac{\sqrt{2}}{2}(\cos 2A-\sin 2A)$.
37. $\cos 2A$. 38. $\frac{1}{2}[1-\sqrt{3}]$. 39. $2\cos A\cos B$.
40. $-2\sin A\sin B$. 41. $2\sin A\cos B$. 42. $2\cos A\sin B$.
43. $\frac{\cot A\cot B-1}{\cot A+\cot B}$; $\frac{\cot A\cot B+1}{\cot A-\cot B}$.

EXERCISE 6.2 (page 164)

1. $2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$; $-2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$.
2. (a) $2\sin 4x\cos 2x$. (b) $2\cos 4x\cos 2x$. (c) $-2\cos\frac{3A}{2}\sin\frac{A}{2}$.
(d) $2\sin\frac{3A}{2}\sin\frac{A}{2}$. (e) $2\sin 4\theta\cos\theta$. (f) $2\cos 4\theta\cos\theta$.
(g) $2\cos\frac{2A+B}{2}\sin\frac{A}{2}$. (h) $-2\sin A\sin\frac{A}{2}$.
3. (a) o. (b) $-\sqrt{3}$. (c) 1. (d) $\sqrt{3}$.

EXERCISE 6.3 (page 167)

1. 11°. 2. 14°. 3. 59°. 4. 80°. 5. 45°.
6. 80°. 7. $\frac{5}{3}$. 8. $\pm 4\sqrt{3}-8$. 9. $2y=x+11$.

EXERCISE 6.4 (page 170)

1. $\cos^2 2A-\sin^2 2A$. 2. $2\cos 2A\sin 2A$. 3. $\frac{2\tan 2A}{1-\tan^2 2A}$.
4. $2\sin 6A\cos 6A$. 5. $\cos^2 6A-\sin^2 6A$. 6. $2\cos 5\pi\sin 5\pi$.
7. $2\cos^2\frac{\pi}{8}-1$. 8. $2\cos^2\frac{7\pi}{4}-1$. 9. $\frac{2\tan 120°}{1-\tan^2 120°}$.

10. $\frac{120}{169}$; $\frac{119}{169}$; $\frac{120}{119}$; 1st quadrant.
11. $\frac{4}{5}$; $-\frac{1}{5}$; -4; 2nd quadrant.
12. $4\cos^3 A - 3\cos A$.
13. $-\frac{44}{125}$.
14. (a) $\dfrac{1-\tan^2 A}{2\tan A}$. (b) $\dfrac{\cot^2 A - 1}{2\cot A}$.
15. $\dfrac{2}{\sqrt{5}}$; $\dfrac{1}{\sqrt{10}}$.
16. $\dfrac{\sqrt{3}}{2}$, $\frac{1}{2}$.
18. $\pm\sqrt{\dfrac{1+\cos\theta}{2}}$; $\pm\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$.

EXERCISE 6.5 (page 174)

1 and **2.**

	Period	Amplitude	Phase angle
(a)	2π	2	$\frac{\pi}{3}$
(b)	2π	5	$-53°$
(c)	2π	13	$67°$
(d)	2π	$\sqrt{5}$	$27°$
(e)	π	2	
(f)	$\frac{2}{3}$	$\sqrt{5}$	

REVIEW EXERCISE 6 (page 175)

1. 59°. **2.** 82°. **4.** $\dfrac{\sqrt{3}}{2}$.

17.

	Period	Amplitude	Phase angle
(a)	2π	2	30°.
(b)	2π	2	−30°.
(c)	2π	5	−37°.
(d)	2π	$2\sqrt{2}$	45°.

Chapter 7

EXERCISE 7.1 (page 183)

1. (a) (4, −3), (−3, 5), (−1, −3), (4, 4).
(b) (1, −1), (−6, 7), (−4, −1), (1, 6).
(c) (6, −7), (−1, 1), (1, −7), (6, 0).
(d) (0, −10), (−7, −2), (−5, −10), (0, −3).
2. (a) (−3, 0), (−1, 3), (1, 6), (5, 12).
(b) (−4, −4), (−2, −1), (0, 2), (4, 8).
(c) (0, −4), (2, −1), (4, 2), (8, 8).
(d) (2, 5), (4, 8), (6, 11), (10, 17).
3. (a) (−1, −3), (0, 0), (−6, 1), (−7, −2).
(b) (6, −1), (7, 2), (1, 3), (0, 0).
4. (a) (0, 0), (−3, 2), (−2, −4).
(b) (2, 4), (−1, 6), (0, 0).

EXERCISE 7.2 (page 187)

1. $AB = 4$, $BC = 3$, $CA = 5$. **2.** Slope $AC = \frac{3}{4}$.

EXERCISE 7.3 (page 193)

1. $v = 3u + 7.$ 2. $v = -2u + 5.$ 3. $v = 4u.$ 4. $v = 2u.$
5. $v = -u.$ 6. $2v = 3u.$ 7. $2v = 3u.$ 8. $v = 4 - u.$
9. $v = 2 - u.$ 10. $v = 2u.$ 11. $h = 0, k = -3$ or $h = \frac{3}{2}, k = 0.$
12. $h = 0, k = 1$ or $h = 1, k = 5.$ 13. $h = 1, k = 6$ or $h = -1, k = 1.$
14. $h = -1, k = 1$ or $h = 2, k = -3.$ 15. $h = -1, k = -1$ or $h = 5, k = -5.$
16. $h = -2, k = 0$ or $h = 1, k = 2.$ 17. $h = 0, k = -3$ or $h = -2, k = 0.$
18. $h = 0, k = -\frac{3}{\pi}$ or $h = \frac{3}{\sqrt{2}}, k = 0.$ 19. No. 20. Yes.
21. (a) $u + 2v < 5$; yes. (b) $3u + 2v < 16$; yes.
22. If slope of line $= \frac{k}{h}$, then no change in regions occurs for the translation $(x, y) \rightarrow (x + h, y + k).$
23. (a) $x > 1, y < -1.$ (b) $3y - 2x \leq 33, 2x - y < -11.$
(c) $y - x < 3, x + 2y > -9, 3x + 2y < -15.$

EXERCISE 7.4 (page 197)

1. $(0, 0) \rightarrow (-\frac{3}{2}, -\frac{3}{2})$ $(3, 0) \rightarrow (\frac{3}{2}, -\frac{3}{2}).$
$(0, 3) \rightarrow (-\frac{3}{2}, \frac{3}{2})$ $(3, 3) \rightarrow (\frac{3}{2}, \frac{3}{2}).$
2. Centre is $(3, -4)$. Centre in $v - u$ plane is $(0, 0)$ $(x, y) \rightarrow (x - 3, y + 4) = u, v.$
4. $(x, y) \rightarrow (x - 3, y + 2) = (u, v); u^2 + v^2 = 1.$
5. $(x, y) \rightarrow (x - 2, y - 3) = (u, v); u^2 + v^2 = 31.$
6. $(x, y) \rightarrow (x + 3, y - 4) = (u, v); u^2 + v^2 = 49.$
7. $(x, y) \rightarrow (x - 12, y - 5) = (u, v); u^2 + v^2 = 169.$
8. $(x, y) \rightarrow (x + 2\sqrt{2}, y + 1) = (u, v); u^2 + v^2 = 9.$
9. $(x, y) \rightarrow (x - 1), y - 3) = (u, v).$

EXERCISE 7.5 (page 201)

	Curve	Image
1.	Parabola	$y^2 - 4x = 0.$
2.	Hyperbola	$y^2 - 2x^2 = \frac{17}{2}.$
3.	Pair of straight lines	$x^2 - 4y^2 = 0.$
4.	Ellipse	$2x^2 + 3y^2 = 6.$
5.	Hyperbola	$9x^2 - 4y^2 = 118.$
6.	Pair of straight lines	$3x^2 - y^2 = 0.$
7.	Parabola	$x^2 = y.$
8.	Ellipse	$x^2 + 4y^2 = 54.$
9.	Ellipse	$4x^2 + 9y^2 = 36.$
10.	Hyperbola	$9y^2 - 4x^2 = 13.$

EXERCISE 7.6 (page 203)

	Translation	Curve
1.	$(x, y) \to (x+1, y-2)$	Parabola.
2.	$(x, y) \to (x+4, y+\frac{77}{4})$	Parabola.
3.	$(x, y) \to (x-2, y-4)$	Parabola.
4.	$(x, y) \to (x-3, y+\frac{1}{4})$	Parabola.
5.	$(x, y) \to (x+6, y-2)$	Hyperbola.
6.	$(x, y) \to (x+2, y+\frac{3}{4})$	Ellipse.
7.	$(x, y) \to (x+\frac{3}{2}, y-5)$	Ellipse.
8.	$(x, y) \to (x+4, y+7)$	Hyperbola.
9.	$(x, y) \to (x+2, y-3)$	Ellipse.
10.	$(x, y) \to (x-1, y-4)$	Hyperbola.

11. (i) $ab = 0 \Rightarrow$ straight line or parabola. (ii) $ab > 0 \Rightarrow$ ellipse.
(iii) $ab < 0 \Rightarrow$ hyperbola. **12.** $h = \frac{g}{a}$, $k = \frac{f}{b}$.

REVIEW EXERCISE 7 (page 204)

1. $A'(4, 2)$, $B'(3, -3)$, $C'(-1, -23)$.
2. Triangle. $A'(-1, 2)$, $B'(-4, -2)$, $C'(2, -2)$.
3. $A'(0, 0)$, $B'(-1, 1-\sqrt{3})$, $C'(3, -2\sqrt{3})$.
5. $3v = 2u - 30$.
6. $h = -\frac{3}{4}$, $k = 0$ **7.** $h = -2$, $k = -1$ **8.** $h = -3$, $k = -4$.
9. $h = 0$, $k = -1$.
10. $3u - 5v < 14$.
11. (b) $3v < u + 10$; yes.
12. $4u^2 - 24u + 4v^2 + 16v + 27 = 0$.
13. $(x, y) \to (x-5, y+3)$; $x^2 + y^2 = 12$.
14. $(x, y) \to (x-\frac{1}{2}, y+\frac{3}{2})$; $4x^2 + 4y^2 = 25$.
15. $9y^2 - 4x^2 = 26$; hyperbola with foci on y-axis.
16. $x^2 + 4y^2 = \frac{25}{4}$; ellipse with foci on x-axis.
17. $(x, y) \to (x+3, y-4)$; parabola opening to the left.
18. $(x, y) \to (x-5, y-1)$; ellipse with foci on x-axis.
19. $(x, y) \to (x-3, y+5)$; ellipse with foci on y-axis.
20. $(x, y) \to (x-1, y+2)$; ellipse with foci on y-axis.
21. $(x, y) \to (x-\frac{3}{2}, y+\frac{1}{2})$; parabola opening upwards.

Chapter 8

EXERCISE 8.1 (page 211)

1. (a) A: $\left(\frac{\sqrt{2}}{2}, \frac{7\sqrt{2}}{2}\right)$ $\theta = 45°$. B: $(-2\sqrt{2}, \sqrt{2})$ $\theta = 45°$.
C: $\left(\frac{\sqrt{2}}{2}, -\frac{3\sqrt{2}}{2}\right)$ $\theta = 45°$. D: $(3\sqrt{2}, \sqrt{2})$ $\theta = 45°$.

(b) $\left(-\frac{7\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}\right)$ $(-\sqrt{2}, -2\sqrt{2})$ $\left(\frac{3\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$ $(-\sqrt{2}, 3\sqrt{2})$ $\theta = 135°$.

(c) $\left(2 - \frac{3}{2}\sqrt{3}, 2\sqrt{3} + \frac{3}{2}\right)$ $\left(-\frac{1}{2} - \frac{3}{2}\sqrt{3}, -\frac{\sqrt{3}}{2} + \frac{3}{2}\right)$ $\left(-\frac{1}{2} + \sqrt{3}, -\frac{1}{2}\sqrt{3} - 1\right)$ $(2 + \sqrt{3}, 2\sqrt{3} - 1)$ $\theta = 60°$.

(d) $(-3, 4)$ $(-3, -1)$ $(2, -1)$ $(+2, 4)$ $\theta = 90°$.

2. (a) $\left(\frac{-2\sqrt{3} + 1}{2}, \frac{-2 - \sqrt{3}}{2}\right)$, $(0, 0)$, $\left(\frac{2\sqrt{3} - 1}{2}, \frac{2 + \sqrt{3}}{2}\right)$, $\left(\frac{6\sqrt{3} - 3}{2}, \frac{6 + 3\sqrt{3}}{2}\right)$, $\theta = 30°$.

(b) $\left(\frac{-3}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $(0, 0)$, $\left(\frac{3}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$, $\left(\frac{9}{\sqrt{2}}, \frac{-3}{\sqrt{2}}\right)$, $\theta = -45°$.

(c) $(2, 1)$ $(0, 0)$, $(2, 1)$ $(6, 3)$ $\theta = 180°$.

(d) $\left(\frac{2 + \sqrt{3}}{2}, \frac{-2\sqrt{3} + 1}{2}\right)$, $(0, 0)$, $\left(\frac{-2 - \sqrt{3}}{2}, \frac{2\sqrt{3} - 1}{2}\right)$, $\left(\frac{-6 - 3\sqrt{3}}{2}, \frac{6\sqrt{3} - 3}{2}\right)$, $\theta = 120°$. Distance and angles are preserved.

3. $AB = \sqrt{13}$, $AD = \sqrt{17}$

4. $\theta = 60°$ $(x, y) \to \left(\frac{x}{2} - \frac{y\sqrt{3}}{2}, \frac{x\sqrt{3}}{2} + \frac{y}{2}\right)$.

$\theta = 120°$ $(x, y) \to \left(-\frac{x}{2} - \frac{y\sqrt{3}}{2}, \frac{x\sqrt{3}}{2} - \frac{y}{2}\right)$.

EXERCISE 8.2 (page 216)

2. (a) $(0, -2)$, $(4, -2)$, $(4, 2)$, $(0, 2)$.

3. (a) $(-1, \sqrt{3})$, $(-2, 0)$, $(-1, -\sqrt{3})$, $(1, -\sqrt{3})$, $(2, 0)$, $(1, \sqrt{3})$, $\theta = 120°$.

(b) $\theta = 60°$ $(x, y) \to \left(\frac{x}{2} - \frac{y\sqrt{3}}{2}, \frac{x\sqrt{3}}{2} + \frac{y}{2}\right)$.

$\theta = 180°$ $(x, y) \to (-x, -y)$.

$\theta = 240°$ $(x, y) \to \left(-\frac{x}{2} + \frac{y\sqrt{3}}{2}, -\frac{x\sqrt{3}}{2} - \frac{y}{2}\right)$.

$\theta = 300°$ $(x, y) \to \left(\frac{x}{2} + \frac{y\sqrt{3}}{2}, -\frac{x\sqrt{3}}{2} + \frac{y}{2}\right)$.

4. $u = \frac{x - y}{\sqrt{2}}$, $v = \frac{x + y}{\sqrt{2}}$, $x = y$. $u = 0$, $v = \frac{2x}{\sqrt{2}} = \sqrt{2}x$. $u = 0$.

5. $y = \sqrt{3}x$. $x = \frac{\sqrt{3}u + v}{2}$. $y = \frac{-u + \sqrt{3}v}{2}$. $\frac{-u + \sqrt{3}v}{2} = \sqrt{3}\left(\frac{\sqrt{3}u + 5}{2}\right)$. $-u + \sqrt{3}v = 3u + \sqrt{3}v$. $u = 0$.

6. $x = y$.

7. $x = \frac{u + v}{\sqrt{2}}$, $\frac{-u + v}{\sqrt{2}} - \left(\frac{u + v}{\sqrt{2}}\right) = 4$. $y = \frac{-u + v}{\sqrt{2}}$, $2u = -4\sqrt{2}$, $u = -2\sqrt{2}$.

8. $x = \frac{-u - v}{\sqrt{2}}$, $y = \frac{u - v}{\sqrt{2}}$, $-\sqrt{2}v + \sqrt{2} = 0$. $v = 1$.

9. $x = \frac{3u + 4v}{5}$, $\frac{-4u + 3v}{5} = 4$. $y = \frac{-4u + 3v}{5}$, $-4u + 3v = 20$. $4u - 3v + 20 = 0$.

10. $x = \dfrac{3u - 4v}{5}$, $y = \dfrac{4u + 3v}{5}$, $4\left(\dfrac{3u - 4v}{3}\right) + 3\left(\dfrac{4u + 3v}{5}\right) = 5;$
$3\left(\dfrac{3u - 4v}{5}\right) - 4\left(\dfrac{4u + 3v}{5}\right) = -5.$ $12u - 16v + 12u + 9v = 25;$
$9u - 12v - (16u + 12v) = -25.$ $24u - 7v = 25;$ $+7u + 24v = +25.$

11. $u^2 + v^2 = 4.$ **12.** $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$, $4u^2 + v^2 = 16.$

13. $x = \dfrac{u - v}{\sqrt{2}}$, $y = \dfrac{u + v}{\sqrt{2}}$. $v^2 = -\frac{16}{3}.$
There exists no real v satisfying this last equation; therefore the image is the null set. Note that $3x^2 - 6xy + 3y^2 = 3(x - y)^2 = -32$ has no solution for x and y; the original curve is also the null set.

14. $x = \dfrac{\sqrt{3}u - v}{2}$, $y = \dfrac{u + \sqrt{2}v}{2}$. $\dfrac{2u^2}{3} - 2v^2 = 1.$
$a^2 = \frac{3}{2}$, $b^2 = \frac{1}{2}$, $c^2 = 2 = a^2 + b^2$, $e = \dfrac{c}{a} = \dfrac{2}{\sqrt{3}} = \frac{2}{3}\sqrt{3}$. Foci $(\pm\sqrt{2}, 0)$.

15. $x = \dfrac{u - v}{\sqrt{2}}$, $y = \dfrac{u + v}{\sqrt{2}}$. $v^2 = \dfrac{1}{\sqrt{2}}u.$

16. $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$. $uv = 2.$

17. $xy = 4.$ $x = \dfrac{u - v}{\sqrt{2}}$, $y = \dfrac{u + v}{\sqrt{2}}$. $\dfrac{u^2}{8} - \dfrac{v^2}{8} = 1.$
$c^2 = a^2 + b^2 = 128 = c = 8\sqrt{2}$. Foci are $(\pm 8\sqrt{2}, 0)$, vertices are $(\pm 2\sqrt{2}, 0)$.

18. $\dfrac{x^2}{16} + \dfrac{y^2}{9} = 1$, $x = \dfrac{4u + 3v}{5}$, $y = \dfrac{-3u + 4v}{5}$. $288u^2 + 337v^2 - 168uv = 3600.$

19. $x = \dfrac{\sqrt{3}u + v}{2}$, $y = \dfrac{-u + \sqrt{3}v}{2}$. $3u^2 + v^2 + 2\sqrt{3}uv = -4u + 4\sqrt{3}v.$

20. $(3x - 4y)^2 = 5(4x + 3y)$. $3x - 4y = 5u$, $4x + 3y = 5v$. $(5u)^2 = 5(5v)$.
$u^2 = v.$

21. $x = \dfrac{2u + 3v}{\sqrt{13}}$; so $\dfrac{2u + 3v}{\sqrt{13}} < 2$ or $2u + 3v < 2\sqrt{13}$.
This region is that to the left of (or, equivalently, "under") the line $2u + 3v = 2\sqrt{13}$.

22. $y < x \le y + 1$ $x = \dfrac{-5u + 12v}{13}$, $y = \dfrac{-12u - 5v}{13}$.
$\dfrac{-12u - 5v}{13} < \dfrac{-5u + 12v}{13} \le \dfrac{-12u - 5v}{13} + 1$ $0 < 7u + 17v \le 13.$
This gives the area between $7u + 17v = 0$ and $7u + 17v = 13$, including the line $7u + 17v = 13$.

23. $-y < x + 1 < y < 1$ $x = \dfrac{-u - \sqrt{7}v}{2\sqrt{2}}$, $y = \dfrac{+\sqrt{7}u - v}{2\sqrt{2}}$,
$v - \sqrt{7}u < -u - \sqrt{7}v + 2\sqrt{2} < \sqrt{7}u - v < 2\sqrt{2}.$
The result is a triangle congruent to the original triangle.

EXERCISE 8.3 (page 224)

1. $\theta = 45°$. $(x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right)$. $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$.
$(x - y)^2 = 12(x + y)$. $(\sqrt{2}u)^2 = 12(\sqrt{2}v)$, $2u^2 = 12\sqrt{2}u$, $u^2 = 6\sqrt{2}v$.

2. $\theta = 60°$. $(x, y) \to \left(\dfrac{x - \sqrt{3}y}{2}, \dfrac{\sqrt{3}x + y}{2}\right)$.

$x = \dfrac{u + \sqrt{3}v}{2}$, $y = \dfrac{-\sqrt{3}u + v}{2}$. $\dfrac{v^2}{2} - \dfrac{u^2}{2} = 1$.

3. $\tan 2\theta = \dfrac{4\sqrt{3}}{-4} = -\sqrt{3}$, $\theta = -30°$, $(x, y) \to \left(\dfrac{\sqrt{3}x + y}{2}, \dfrac{x - \sqrt{3}y}{2}\right)$.

$x = \dfrac{\sqrt{3}u - v}{2}$, $y = \dfrac{+u + \sqrt{3}v}{2}$. $\dfrac{v^2}{3} - \dfrac{u^2}{9} = 1$.

4. $\theta = 45°$, $(x, y) \to \left(\dfrac{x + y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right)$. $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$.

$\dfrac{u^2}{25} + \dfrac{v^2}{9} = 1$.

5. $\theta = 45°$, $(x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right)$, $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$. $\dfrac{3u^2}{16} - \dfrac{v^2}{16} = 1$.

6. $\theta = 45°$. $(x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right)$. $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$.

$v^2 - u^2 = 18$.

7. $\theta = 45°$. $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$. $\dfrac{4v^2}{25} - \dfrac{2u^2}{25} = 1$.

8. $\theta = 45°$, $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$. $\dfrac{u^2}{9} + \dfrac{v^2}{16} = 1$.

9. $\theta = 45°$, $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$. $u = \dfrac{x - y}{\sqrt{2}}$, $v = \dfrac{x + y}{\sqrt{2}}$.

$2u^2 - 2\sqrt{2}v + 1 = 0$.

10. $\tan 2\theta = \dfrac{2\sqrt{3}}{2} = \sqrt{3}$, $\theta = 30°$. $(x, y) \to \left(\dfrac{\sqrt{3}x - y}{2}, \dfrac{x + \sqrt{3}y}{2}\right)$.

$x = \dfrac{\sqrt{3}u + v}{2}$, $y = \dfrac{-u + \sqrt{3}v}{2}$. $v^2 + u + \sqrt{3}v = 0$.

11. $\theta = 45°$. $(x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right)$ $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$

$\dfrac{(u - \frac{9}{\sqrt{2}})^2}{39} - \dfrac{(v - \frac{1}{\sqrt{2}})^2}{13} = 1$. $\dfrac{s^2}{39} - \dfrac{t^2}{13} = 1$.

12. $\theta = 45°$. $(x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right)$, $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$.

$\dfrac{32(u + \frac{9}{4}\sqrt{2})^2}{3} + \dfrac{128(v - \frac{3}{16}\sqrt{2})^2}{3} = -1$.

This equation has no solution; the original and image curves are both the empty set.

13. $\theta = 45°$. $(x, y) \to \left(\dfrac{x - y}{\sqrt{2}}, \dfrac{x + y}{\sqrt{2}}\right)$. $x = \dfrac{u + v}{\sqrt{2}}$, $y = \dfrac{-u + v}{\sqrt{2}}$.

$(x - y)^2 + 5x = 0$. $\left(v + \dfrac{5}{4\sqrt{2}}\right)^2 + \dfrac{5}{2\sqrt{2}}u = \frac{25}{32}$.

$\left(v + \dfrac{5}{4\sqrt{2}}\right)^2 = -\dfrac{5}{2\sqrt{2}}\left(u - \dfrac{5\sqrt{2}}{16}\right)$. $S^2 = -\dfrac{5}{2\sqrt{2}}T$.

14. $(x + 3)^2 - 3y + 6 = 9$. $(x + 3)^2 = 3(y + 1)$. $s^2 = 3t$.

15. $(\sqrt{3}x - y)^2 + 4(x - \sqrt{3}y) = 0.\quad \theta = 30°.\quad (x, y) \to \left(\frac{\sqrt{3}x - y}{2}, \frac{x + \sqrt{3}y}{2}\right).$
$x = \frac{\sqrt{3}u + v}{2},\ y = \frac{-u + \sqrt{3}v}{2}.\quad u^2 + \sqrt{3}u = v.$
$\left(u + \frac{\sqrt{3}}{2}\right)^2 = v + \frac{3}{4}.\quad S^2 = T.$

16. (a) $x = \frac{u + v}{\sqrt{2}},\ y = -\frac{u + v}{\sqrt{2}}.\quad (u, v) = \left(\frac{9}{\sqrt{2}} + \sqrt{39}, \frac{1}{\sqrt{2}}\right)\left(\frac{9}{\sqrt{2}} - \sqrt{39}, \frac{1}{\sqrt{2}}\right).$
$(x, y) = \left(\frac{10}{\sqrt{2}} + \sqrt{39}, -\frac{8}{\sqrt{2}} - \sqrt{39}\right)\left(\frac{10}{\sqrt{2}} - \sqrt{39}, -\frac{8}{\sqrt{2}} + \sqrt{39}\right).$
(c) $x = \frac{u + v}{\sqrt{2}},\ y = \frac{-u + v}{\sqrt{2}}.\quad (u, v) = \left(-\frac{5\sqrt{2}}{8}, \frac{5\sqrt{2}}{16}\right).$
$(x, y) = \left(\frac{5\sqrt{2}}{16}, \frac{15\sqrt{2}}{16}\right).$
(d) $(x, y) = (-3, -1).$
(e) $(u, v) = \left(-\frac{\sqrt{3}}{2}, -\frac{3}{4}\right).\quad x = \frac{\sqrt{3}u + v}{2},\ y = \frac{-u + \sqrt{3}v}{2}.$
$(x, y) = \left(-\frac{9}{8}, -\frac{\sqrt{3}}{8}\right).$

17. (b) $\tan 2\theta = \frac{2h}{b - a}.$

18. $\tan 2\theta = \sqrt{3},\ 2\theta = 60°,\ \theta = 30°.\quad \sin\theta = \frac{1}{2},\ \cos\theta = \frac{1}{2}\sqrt{3}.$

19. $2\theta = 90°,\ \theta = 45°.$

REVIEW EXERCISE 8 (page 225)

1. (a) $\left(\frac{2\sqrt{3} + 1}{2}, \frac{2 - \sqrt{3}}{2}\right), \left(\frac{3\sqrt{3} - 4}{2}, \frac{3 + 4\sqrt{3}}{2}\right), \left(\frac{-5}{2}, \frac{5\sqrt{3}}{2}\right),$
$\left(\frac{1 - \sqrt{3}}{2}, \frac{-1 - \sqrt{3}}{2}\right),\ \theta = -30°.$
(b) $(-2, +1), (-3, -4), (0, -5), (1, +1).\quad \theta = 180°.$
(c) $(\sqrt{5}, 0)\left(\frac{2}{\sqrt{5}}, \frac{11}{\sqrt{5}}\right)(-\sqrt{5}, 2\sqrt{5})\left(\frac{-1}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right).$
$\sin\theta = -\frac{1}{\sqrt{5}} = -\frac{1}{5}\sqrt{5}.\quad \sin\theta = -(.2)(2.236) = -.447.\quad \theta = -27°.$

2. $(x, y) \to \left(\frac{x - \sqrt{3}y}{2}, \frac{\sqrt{3}x + y}{2}\right).\quad (0, 0) \to (0, 0).\quad (5, 0) \to (\frac{5}{2}, \frac{5}{2}\sqrt{3}).$
$(4, 3) \to \left(\frac{4 - 3\sqrt{3}}{2}, \frac{4\sqrt{3} + 3}{2}\right).$

3. $(x, y) \to \left(\frac{x + y}{\sqrt{2}}, \frac{-x + y}{\sqrt{2}}\right) = (u, v).\quad x = \frac{u - v}{\sqrt{2}},\ y = \frac{u + v}{\sqrt{2}}.$
$\frac{u + v}{\sqrt{2}} = \frac{u - v}{\sqrt{2}};\ v = 0.$

4. $v = -\frac{5}{\sqrt{2}}.$

5. $6u - 8v = -15.$

6. $u^2 + v^2 = 9.$

7. $\frac{v^2}{16} - \frac{u^2}{16} = 1.$

8. $\frac{v^2}{4} - \frac{u^2}{25} = 1.$

9. $\theta = 45°$. $(x, y) \to \left(\dfrac{x-y}{\sqrt{2}}, \dfrac{x+y}{\sqrt{2}}\right)$. $x = \dfrac{u+v}{\sqrt{2}}, y = \dfrac{-u+v}{\sqrt{2}}$. $\dfrac{u^2}{2} + \dfrac{v^2}{4} = 1$.
$(0, \sqrt{2})\frac{1}{2}(0)^2 + \frac{1}{4}(\sqrt{2})^2 = \frac{1}{2} < 1$, inside.
$\left(-\dfrac{3\sqrt{2}}{2}, \dfrac{\sqrt{2}}{2}\right)$ $\frac{1}{2}(\frac{9}{2}) + \frac{1}{4}(\frac{1}{2}) = \frac{9}{4} + \frac{1}{8} > 1$, outside.

10. $\tan 2\theta = \dfrac{24}{-7}$; $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2T}{1-T^2} = \dfrac{-24}{7}$.
$T = \dfrac{7 \pm \sqrt{49 + 4(144)}}{24} = \dfrac{7 \pm 25}{24}$. $\tan\theta = \frac{4}{3}$ or $-\frac{3}{4}$.
Let $0 < 2\theta < 180°$. Then, $0 < \theta < 90°$.
Then, $\tan\theta = \frac{4}{3}, \sin\theta = \frac{4}{5}, \cos\theta = \frac{3}{5}$.
$(x, y) \to \left(\dfrac{3x-4y}{5}, \dfrac{4x+3y}{5}\right), x = \dfrac{3u+4v}{5}, y = \dfrac{-4u+3v}{5}$.
$x = \dfrac{3u+4v}{5}, y = \dfrac{-4u+3v}{5}$. $\dfrac{u^2}{10} + \dfrac{3v^2}{20} = 7$.
$(-\frac{9}{5}, -\frac{13}{5}), \dfrac{162+507}{500} > 1$, outside. $\left(-\frac{4}{5}, \dfrac{-2}{5}\right), \dfrac{242+12}{500} < 1$, inside.

11. $\theta = 45°, (x, y) \to \left(\dfrac{x-y}{\sqrt{2}}, \dfrac{x+y}{\sqrt{2}}\right)$.

12. $\theta = 45°, (x, y) \to \left(\dfrac{x-y}{\sqrt{2}}, \dfrac{x+y}{\sqrt{2}}\right)$.

13. $\tan 2\theta = \frac{12}{5}, \frac{12}{5} = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2T}{1-T^2}$. Choose $0 < 2\theta < 180°$. So $0 < \theta < 90°$.
$T = \dfrac{-5 \pm \sqrt{25 + 144}}{12} = \dfrac{-5 \pm 13}{12} = \frac{2}{3}$ or $-\frac{3}{2}$.
$\tan\theta = \frac{2}{3} = .66666 \cdots, (x, y) \to \left(\dfrac{3x-2y}{\sqrt{13}}, \dfrac{2x+3y}{\sqrt{13}}\right)$.

14. $\tan 2\theta = \dfrac{-20}{21} \dfrac{2T}{1-T^2} - 20 + 20T^2 = 42T$. Let $0 < 2\theta < 180°$.
So $0 < \theta < 90°$. $20T^2 - 42T - 20 = 0$. $10T^2 - 21T - 10 = 0$.
$T = \dfrac{21 \pm \sqrt{441 + 400}}{20} = \dfrac{21 \pm 28}{20} = \frac{49}{20}, \dfrac{-7}{20}$. $\tan\theta = \frac{49}{20}$.
$(x, y) \to \left(\dfrac{20x - 49y}{\sqrt{2801}}, \dfrac{49x+20y}{\sqrt{2801}}\right)$.

16. (a) $\tan 2\theta = \dfrac{24}{-7}$, $\tan\theta = \frac{4}{3}$ or $-\frac{3}{4}$ from question (10).

Choose $\tan\theta = \frac{4}{3}$ and $0 < \theta < 90°$.
$(x, y) = \left(\dfrac{3x-4y}{5}, \dfrac{4x-3y}{5}\right)$. $x = \dfrac{3U+4V}{5}, y = \dfrac{-4U+3V}{5}$.
$10U^2 + 28U - 15V^2 - 6V - 11 = 0$. $\dfrac{(U+\frac{7}{5})^2}{3} - \dfrac{(V+\frac{1}{5})^2}{2} = 1$.
$\dfrac{s^2}{3} - \dfrac{t^2}{2} = 1$. Hyperbola.

(b) $(x, y) \to (x+1, y-1), 6u^2 + 24uv - v^2 + 53 = 0$, same final curve as in (a).
(c) Hyperbola.

17. $a = \sqrt{3}, b = \sqrt{2}, c^2 = a^2 + b^{\hat{}} = 5, c = \sqrt{5}$. Let $0 < 2\theta < 180°$.
Foci: $(\pm\sqrt{5}, 0)$.

18. $\tan 2\theta = \dfrac{3}{-4} = \dfrac{2T}{1 - T^2}$. $3 - 3T^2 = -8T$. $3T^2 - 8T - 3 = 0$.

$T = \dfrac{8 \pm \sqrt{64 + 36}}{6} = \dfrac{8 \pm 10}{6} = 3$ or $-\frac{1}{3}$. $\tan \theta = 3$.

$(x, y) \to \left(\dfrac{x - 3y}{\sqrt{10}}, \dfrac{3x + y}{\sqrt{10}}\right)$, $x = \dfrac{u + 3v}{\sqrt{10}}$, $y = \dfrac{-3u + v}{\sqrt{10}}$.

$\dfrac{u^2}{26} + \dfrac{v^2}{6} = 1$. Ellipse. Foci: $(u, v) = (\sqrt{26}, 0), (-\sqrt{26}, 0)$.

Foci: $(x, y) = (\frac{1}{10}\sqrt{260}, 0 - \frac{3}{10}\sqrt{260}), (-\frac{1}{10}\sqrt{260}, \frac{3}{10}\sqrt{260})$,

Chapter 9

EXERCISE 9.1 (page 231)

1. (a) $(3, -2), (7, -3), (4, 0)$. (d) $x = 0$.

2. (a) $A'(-3, -5), B'(-1, -2), C'(-8, -1)$.

3. (a) $A'(2, 8), B'(2, 2), C'(-1, 2)$. (b) $x = 3$.

4. (a) $M'(0, 1), N'(1, 2), P'(4, 7)$. (b) $x = y$.

5. (a) $A'(1, -9), B'(3, -6), C'(5, -3), A''(-1, -9), B''(-3, -6), C''(-5, -3)$.
(c) $x = 0, y = 0$. (d) $(x, y) \to (-x, -y)$.

6. (a) $A''(-3, -9), B''(-5, -6), C''(-7, -3)$. (b) $x = -1$.
(c) $(x, y) \to (-x, -2, -y)$. (d) Point is $(-1, 0)$.

7. (a) $V(W(3, 1)) = (-2, -4), V(W(-6, -1)) = (7, -2)$,
$V(W(2, -4)) = (-1, 1), W(V3, 1)) = (-4, 2), W(V(-6, -1) = (5, 4)$,
$W(V2, -4) = (-3, 7)$.

EXERCISE 9.2 (page 236)

1. R_1: $-2x + 3y = -6$. R_2: $+2x + 3y = +6$. **2.** Image: $(-2, \frac{7}{3}) \to (2, -\frac{7}{3})$.

3. Image: $x^2 + y^2 + 10y = 0$. **4.** $-2x + 3y < -6$.

5. $x^2 + y^2 \leq 16, 3x - 5y \geq 15$. **6.** $x + y < -2, x + y > -6$.

7. (a) (i) $x^2 - y^2 = 16$. (ii) $x^2 - y^2 = 16$. (iii) $x^2 - y^2 = 16$.
(b) (i) $xy = -8$. (ii) $xy = -8$. (iii) $xy = 8$. (c) Two hyperbolas.

8. $\dfrac{(x - 4)^2}{16} + \dfrac{y^2}{9} = 1$. **9.** (a) $y^2 = -4x$. (c) $x = 1$.

10. R_1: $\dfrac{(y - 12)^2}{144} + \dfrac{x^2}{25} = 1$. R_2: $\dfrac{(x + 12)^2}{144} + \dfrac{y^2}{25} = 1$. **11.** $y = \dfrac{-6}{x^2 + 2}$.

12. $y < \dfrac{-6}{x^2 + 2}$, $x^2 + y^2 < 9$.

EXERCISE 9.3 (page 241)

1. (a) $A'(4, 4), B'(-4, 4), C'(-4, -4), D'(4, -4)$.

2. (a) $A'(28, 36), B'(20, 36), C'(20, 28), D'(28, 28)$.

3. (a) $A'(2, \frac{1}{2}), B'(-2, \frac{1}{2}), C'(-2, -\frac{1}{2}), D'(2, -\frac{1}{2})$. (c) Rectangle.

4. (a) $A'(\sqrt{3}, 0), B'(-\sqrt{3}, 0), C'(0, -3)$.

5. (a) $A'(2,0)$. $B'(-2,0)$, $C'\left(0, -\dfrac{\sqrt{3}}{3}\right)$.
6. $M'(6,6)$, $O'(0,0)$, $N'(-9,9)$, $P'(-9,-6)$, $Q'(6,-6)$.
7. (a) $M'(6,4)$, $O'(0,0)$, $N'(-9,-6)$, $P'(-9,-6)$, $Q'(6,-4)$.

EXERCISE 9.4 (page 246)

1. (a) $u + v + 2 = 0$. $2s + t + 4 = 0$. 2. (a) $u^2 + v^2 = 9$, $\dfrac{s^2}{9} + \dfrac{t^2}{25} = 1$.
3. $x^2 - 6x + y^2 + 18y = -9$, $(3, -9)$, $(3, -9)$.
4. (a) $4u^2 = -5v$. (b) Original: $F(0, -\frac{1}{4})$. Image: $F(0, -\frac{5}{16})$.
5. $\dfrac{u^2}{49} + \dfrac{v^2}{49} = 1$ or $(u^2 + v^2 = 49)$. 6. $\dfrac{(S+8)^2}{49} + T - \frac{49}{6} = 1$.
7. $D_1 \dfrac{u^2}{9} - \dfrac{v^2}{4} = 1$. $D_2\, m^2 - n^2 = 1$.
8. (a) $\dfrac{u^2}{25} - \dfrac{v^2}{49} = -1$. (b) $\dfrac{(u+8)^2}{100} - \dfrac{(v - \frac{3}{7})^2}{1} = -1$.
9. (a) $\dfrac{x^2}{2} + \dfrac{y^2}{10} = 1$. 10. (b) $\dfrac{15u^2}{8} - \dfrac{5v^2}{4} = 1$. (c) Hyperbola $c = \dfrac{\sqrt{6}}{2}$.
   (d) $u^2 - v^2 = \frac{4}{15}$ is a rectangular hyperbola $c = \sqrt{2}$. 11. $v = \dfrac{5}{4u^2 + 1}$.

REVIEW EXERCISE 9 (page 249)

1. (a) $A'(-4,1)$, $B'(-6,5)$, $C'(-9,-1)$.
2. (a) $y = -|x|$. (c) $(4,4) \to (4,-4)$, $(0,0) \to (0,0)$, $(-2,2) \to (-2,-2)$.
3. $P'(-1,-7)$, $(0,-5)$, $(6,-9)$ axis: $y = 1$.
4. $A'(-2,1)$, $B'(-1,3)$, $C'(-6,5)$.
5. $x + y + 1 < 0$; $x - 5y - 5 > 0$; $x < 5$. 6. $4x^2 + y^2 \leq 36$; $2x + y \leq -6$.
7. $9x^2 - 16y^2 - 256y = 1168$, axis: $y = -4$.
9. $A'(0,6)$, $B'(0,-6)$, $C'(9,-6)$, $D'(9,6)$.
10. (a) $A'(0,8)$, $B'(0,-8)$, $C'(3,-8)$, $D'(3,8)$.
11. $A'(-\frac{9}{2}, 6)$, $B'(-\frac{9}{2}, -6)$, $C'(-\frac{9}{2}, -6)$, $D'(\frac{9}{2}, 6)$.
12. $P'(15,0)$, $Q'(15,-18)$, $R'(27,-9)$. 13. $P'(\frac{5}{2}, 0)$, $Q'(\frac{5}{2}, -2)$, $R'(\frac{9}{2}, -1)$.
14. D_1: $4u^2 + 25v^2 = 25$. D_2: $v^2 + v^2 = 1$. 15. $u^2 - v^2 = -100$.
16. $4u^2 - 9v^2 = -25$.
17. $(x,y) \to \left(\dfrac{3x - y}{\sqrt{10}}, \dfrac{x + 3y}{\sqrt{10}}\right)$, $6x^2 + y^2 = 9$, ellipse with foci on y-axis, reflection is $11x^2 + 6xy + 3y^2 - 18 = 0$.

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